ECE 603 - Probability and Random Processes, Fall 2016

Homework #4

Due: 10/14/16, in class

1. The cumulative distribution function of a random variable X is given as

$$F_X(x) = \begin{cases} 0, & \text{for } x < 0 \\ \frac{x}{3}, & \text{for } 0 \leq x < 1 \\ \frac{(x+1)}{3}, & \text{for } 1 \leq x < 2 \\ 1 & \text{for } 2 \leq x \end{cases}.$$

(a) Find the following probabilities:

- $P(X < 1)$.
- $P(1 \leq X \leq 1.5)$.
- $P(X = 1)$.

(b) Find and sketch the probability density function $f_X(x)$ of X.

(c) Find $E[X]$ and Var$[X]$.

2. The probability density function of a random variable X is given by $f_X(x)$, where:

$$f_X(x) = \begin{cases} cx^2, & -2 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find the value of the constant c.

(b) Find the probability that $X^2 \geq 1$.

(c) Find the probability that $X - 1 \geq -\frac{1}{4}$.

(d) Let the random variable Y be defined by:

$$Y = \begin{cases} -X, & X \leq 0 \\ 0, & X \geq 0 \end{cases}$$

Find the probability density function of Y.

3. Consider the probability space $(S, \mathcal{A}, P)$, with $S = [0, 1]$ and $\mathcal{A} = \mathcal{B}$ (restricted to $[0, 1]$,of course), and $P(\cdot)$ defined as follows:

$$P((a, b)) = \begin{cases} c \cdot ((b - a) - \frac{1}{2}(b^2 - a^2)), & 0 \leq a < b \leq \frac{1}{2} \\ c \cdot ((b - a) - \frac{1}{2}(b^2 - a^2)) & \frac{1}{2} \leq a < b \leq 1 \\ \frac{1}{2} + c \cdot ((b - a) - \frac{1}{2}(b^2 - a^2)), & a < \frac{1}{2} < b \leq 1 \end{cases}$$

(a) Find c.

(b) Let ω be the outcome of the experiment. Since ω is a number, it can be treated like a random variable, with cumulative distribution function $F_\omega(x) = P(\omega \leq x)$ and probability density function $f_\omega(x) = \frac{d}{dx} F_\omega(x)$.

- Find the probability density function $f_\omega(x)$.
- Find $E[\omega^2]$.

(c) Let ω be the outcome of the experiment. Consider the mapping $Y : S \to \{\text{Alice, Bob, Carol, Dan}\}$ defined by:

$$Y(\omega) = \begin{cases} \text{Alice}, & 0 \leq \omega < 0.3 \\ \text{Bob}, & 0.3 \leq \omega < 0.6 \\ \text{Carol}, & 0.6 \leq \omega < 0.9 \\ \text{Dan}, & 0.9 \leq \omega \leq 1 \end{cases}$$

Find the probability space $(S_Y, \mathcal{Y}, P_Y)$ for the experiment with outcome Y. Please be *explicit* as this is a very small space! (You can use the abbreviations A, B, C, and D for the names).

(d) Let ω be the outcome of the experiment. Suppose, I define the random variable X by:

$$X(\omega) = \begin{cases} \omega^2, & 0 < \omega \leq \frac{1}{2} \\ \frac{1}{2}, & \omega > \frac{1}{2} \end{cases}$$

Find the probability density function $f_X(x)$ of X.

(e) Let ω be the outcome of the experiment. Suppose I define the random variable Z by $Z = 2e^{-\omega^3} + e^{-\omega^2} + e^{-\omega} + \omega$. Find $P(Z > 6)$.

4. Suppose that I am observing a network connection that is good ("G") with probability 0.9 and bad ("B") with probability 0.1. Let T (in seconds) be the time until the first packet arrives.

   If the connection is good, T has probability density function:

$$f_G(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & \text{else} \end{cases}$$

   If the connection is bad, T has probability density function:

$$f_B(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & \text{else} \end{cases}$$

   (a) What is the probability that the first packet arrives during the first second?

   (b) Given that the first packet arrives during the first second, what is the probability that the link is good?