1. For each of the following systems, determine whether the systems is: (i) linear, (ii) time-invariant, (iii) causal, (iv) stable. If the system is linear and time-invariant (LTI), find the system impulse response $h[n]$.

(a) $y[n] = x[2n]$

(b) $y[n] = x[2 - n]$

(c) $y[n] = \sum_{k=0}^{\infty} (0.5)^k x[n - k]$

(d) $y[n] = (\cos(\pi n)) \cdot x[n]$

(e) $y[n] = (x[n])^2$

(f) $y[n] = x[n^2]$

2. Let $x[n]$ be a purely real sequence. You work in the laboratory to obtain the following information about $x[n]$:

(i) $x[n]$ is a causal sequence.

(ii) If $v[n] = x[n + 2]$, the Fourier transform $V(e^{j\omega})$ of the discrete-time sequence $v[n]$ is purely real.

(iii) \[ \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega = 16 \]

(iv) $x[0] = 2$

(v) \[ \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega} d\omega = 1. \]

(vi) $x[2] > 0$.

Find the discrete-time sequence $x[n]$. 
3. You need to design a system to store an analog signal whose energy is concentrated all around some frequency $f_0$; that is, $X(f)$ is zero except for $f$ in the band $|f| \in [f_0 - \Delta, f_0 + \Delta]$, where $\Delta$ is very small relative to $f_0$. Both $f_0$ and $\Delta$ are known to you as the system designer. Clearly, the Nyquist rate for such a signal would be $f_n = 2 \cdot (f_0 + \Delta)$, but that would potentially require storing a lot of samples per second.

(a) Assume that $f_0 = M2\Delta$ for some integer $M$. Show how you could store this signal (and then recover it) with many fewer samples than the Nyquist rate.

(b) For $f_0 = 100MHz$ and $\Delta = 10kHz$, how much savings in storage can you accomplish with your solution from (a) versus the Nyquist-sampled version?

4. Suppose that I have a discrete-time sequence $x[n]$ with Fourier transform $X(e^{j\omega})$.

(a) Suppose that I form a new sequence $y[n] = x[2n]$. In other words, $y[n]$ takes every other value of $x[n]$ (and discards the rest). Find $Y(e^{j\omega})$ (at least the idea of the picture of it) by supposing that $x[n]$ is an oversampled version of some continuous-time signal.

(b) Suppose that I form a new sequence:

$$z[n] = \begin{cases} x[n/2], & n \text{ even} \\ 0, & n \text{ otherwise} \end{cases}$$

Find $Z(e^{j\omega})$.

(c) Consider again part (b), but now suppose that $x[n]$ is a sampled version of some continuous-time signal. Can you find a filter $H(e^{j\omega})$ such that, when $z(n)$ is input to the filter, the output of the filter is identical to the signal that would have been obtained if $x[n]$ had been sampled twice as fast in the first place?