Homework #3 Solutions

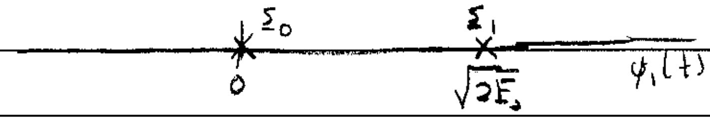
ECE 564/645 — Spring 2014

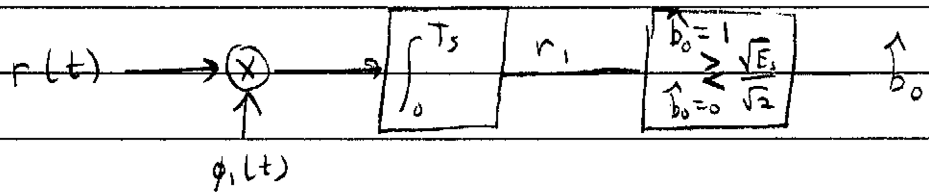
1) (a) Note that $\int_{-\infty}^{\infty} p^2(t)\,dt = 1$.

Thus, let

$$\phi_1(t) = p(t)$$

Signal space:





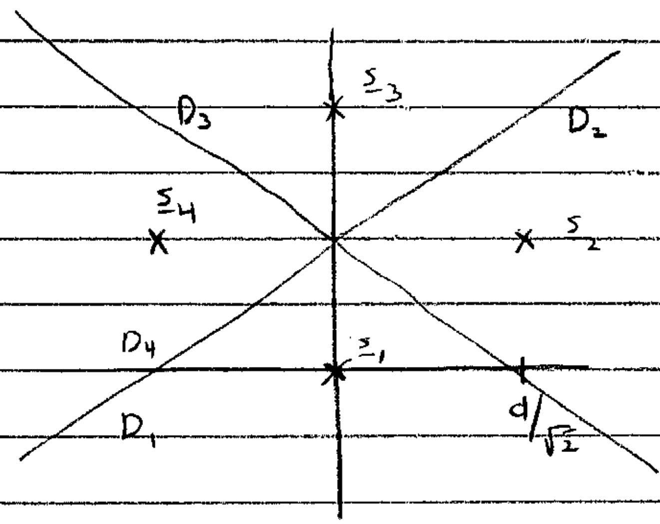
(b) $$P(E) = Q\left(\frac{d}{\sqrt{2N_0}}\right) = Q\left(\frac{\sqrt{2E_s}}{\sqrt{2N_0}}\right) = Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

(c) $$P_{BPSK}(E) = Q\left(\sqrt{\frac{2E_s}{N_0}}\right)$$

This system is 3 dB worse!

$10\log_{10} 2$

2) (a)



(b)

This can be translated to QPSK; thus,

$$P(E) = 2Q\left(\frac{d}{\sqrt{2N_0}}\right) - \left[Q\left(\frac{d}{\sqrt{2N_0}}\right)\right]^2$$

$$E_s = 1/4\,(0 + d^2 + 2d^2 + d^2)$$

$$= d^2$$

$$\Rightarrow P(E) = 2Q\left(\sqrt{E_s/2N_0}\right) - \left[Q\left(\sqrt{E_s/2N_0}\right)\right]^2$$

(c)

$$\text{QPSK: } P_{QPSK}(E) = 2Q\left(\sqrt{E_s/N_0}\right) - \left[Q\left(\sqrt{E_s/N_0}\right)\right]^2$$

This set is 3 dB worse than QPSK

3) (a)

Note $\int_{-\infty}^{\infty} s_2^2(t)\,dt = 1$

Thus, let $\phi_1(t) = s_2(t)$

From class,

$$h(t) = \phi_1(T_s - t)$$

$$= s_2(t) \quad \text{(in this case)}$$

(b)

Suppose $s_1(t)$ is sent. Only noise received:

$$r = \left(\int_{-\infty}^{\infty} h(t-\tau)\, n(\tau)\, d\tau \right) \Big|_{t=T_s}$$

$$= \left(\int_{-\infty}^{\infty} n(t-\tau)\, h(\tau)\, d\tau \right) \Big|_{t=T_s}$$

$$= \int_{-\infty}^{\infty} n(T_s - \tau)\, h(\tau)\, d\tau$$

$$= \int_0^{T_s} {-t}/_{T_s}\; n(T_s - \tau)\, d\tau$$

$$E[r] = \int_0^{T_s} {-t}/_{T_s}\; E[n(T_s - \tau)]\, d\tau$$

$$= 0$$

$$\sigma^2 \triangleq E[r^2] = \int_0^{T_s} \int_0^{T_s} {-t}/_{T_s} \; {-s}/_{T_s}\; E[(n T_s - \tau_1)(n T_s - \tau_2)]\, d\tau_1\, d\tau_2$$

Let's not evaluate this (for now)

Under s_1, $r \sim N(0, \sigma^2)$

Suppose $s_2(t)$ is sent:

$$r = \left(\int_{-\infty}^{\infty} h(t-\tau) * (s_2(\tau) + n(\tau))\, d\tau \right) \Big|_{t=T_s}$$

$$= \underbrace{\left(\int_{-\infty}^{\infty} h(t-\tau)\, s(\tau)\, d\tau \right) \Big|_{t=T_s}}$$

$$+ \left(\int_{-\infty}^{\infty} h(t-\tau) * n(\tau)\, d\tau \right) \Big|_{t=T_s}$$

$$\sim N(0, \sigma^2)$$

$$= \int_{-\infty}^{\infty} h(T_s - \tau)\, s(\tau)\, d\tau$$

$$= \sqrt{\frac{1}{T_s}} \int_0^{T_s} \left(-(T_s - \tau)/T_s \right) d\tau$$

$$= \sqrt{\frac{1}{T_s}} \left(\int_0^{T_s} (-1)\, d\tau + \int_0^{T_s} \tau/T_s\, d\tau \right.$$

$$= \sqrt{\frac{1}{T_s}} \left(-T_s + \tau^2/2T_s \Big|_0^{T_s} \right)$$

$$= \sqrt{\frac{1}{T_s}} \left(-T_s + T_s/2 \right)$$

$$= -\sqrt{T_s}/2$$

Under s_2, $r \sim N\left(-\sqrt{T_s}/2, \sigma^2\right)$

Thus, • choose s_1 if $r \geq -\sqrt{T_s}/4$

• choose s_2 if $r \leq -\sqrt{T_s}/4$

(do not need σ^2 unless I asked for $P(E)$.

4)(a)

Let $\tilde{\phi}_1(t) = \sqrt{2P_c}\cos(2\pi f_c t)$

Need to normalize:

$$\int_0^{T_s} \tilde{\phi}_1^2(t)\,dt = \int_0^{T_s} 2P_c \cos^2(2\pi f_c t)\,dt$$

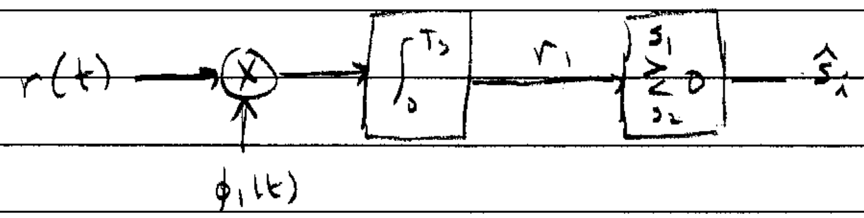
$$= \int_0^{T_s} 2P_c \left(1/2 + 1/2 \cancel{\cos(4\pi f_c t)}^{\text{LPF}} \right) dt$$

$$= P_c T_s$$

$s_1 = \sqrt{P_c T_s}$

$s_2 = \sqrt{P_c T_s}$

Thus, $\phi_1(t) = \dfrac{\tilde{\phi}_1(t)}{\sqrt{P_c T_s}} = \sqrt{\dfrac{2}{T_s}}\cos(2\pi f_c t)$



$$P(E) = Q\left(d/\sqrt{2N_0} \right) = Q\left(\frac{2\sqrt{P_c T_s}}{\sqrt{2N_0}} \right) = Q\left(\sqrt{\frac{2P_c T_s}{N_0}} \right) = Q\left(\sqrt{\frac{2E_s}{N_0}} \right)$$

(of course, this is BPSK!)

(b) under s_1:

$$\int_0^{T_s} \sqrt{2P_c}\cos(2\pi f_c t + \theta_\varepsilon)\sqrt{\frac{2}{T_s}}\cos(2\pi f_c t)\,dt$$

$$= \int_0^{T_s} 2\sqrt{P_c/T_s}\left(\tfrac{1}{2}\cancel{\cos(4\pi f_c t + \theta_\varepsilon)}^{\text{LPF}} + \tfrac{1}{2}\cos\theta_\varepsilon \right) dt$$

$$= \sqrt{P_c T_s}\cos\theta_\varepsilon = \sqrt{E_s}\cos\theta_\varepsilon$$

Under s_2: $-\sqrt{E_s}\cos\theta_\varepsilon$

$$P(E) = Q\left(\frac{d}{\sqrt{2N_0}} \right) = Q\left(\sqrt{\frac{2E_s\cos^2\theta_\varepsilon}{N_0}} \right)$$

5)

Need to compare Choose s_1 if

$$p(\underline{r}|s_1) \gtrless p(\underline{r}|s_2)$$

$$p(r_2|r_1,s_1)p(r_1|s_1) \gtrless p(r_2|r_1,s_2)p(r_1|s_2)$$

$$\cancel{\frac{1}{\sqrt{2\pi N_0/2}}} e^{-\frac{(r_2-(r_1-\sqrt{E_s}))^2}{N_0}} \cancel{\frac{1}{\sqrt{2\pi N_0/2}}} e^{-(r_1-\sqrt{E_s})^2/N_0}$$

$$\gtrless \cancel{\frac{1}{\sqrt{2\pi N_0/2}}} e^{-\frac{(r_2-(r_1+\sqrt{E_s}))^2}{N_0}} \cancel{\frac{1}{\sqrt{2\pi N_0/2}}} e^{-(r_1+\sqrt{E_s})^2/N_0}$$

$\ln(\cdot)$ of both sides, and multiply by $-N_0$:

$$((r_2-r_1)+\sqrt{E_s})^2 + (r_1-\sqrt{E_s})^2 \lessgtr ((r_2-r_1)-\sqrt{E_s})^2 + (r_1+\sqrt{E_s})^2$$

$$\cancel{(r_2-r_1)^2} + 2\sqrt{E_s}(r_2-r_1) + \cancel{E_s} + \cancel{r_1^2} - 2r_1\sqrt{E_s} + \cancel{E_s}$$

$$\gtrless \cancel{(r_2-r_1)^2} - 2\sqrt{E_s}(r_2-r_1) + \cancel{E_s} + \cancel{r_1^2} + 2\sqrt{E_s}r_1 + \cancel{E_s}$$

$$4\sqrt{E_s}\,r_2 \lessgtr 8r_1\sqrt{E_s}$$

$$r_2 \lessgtr 2r_1$$

r_2 tells us about n_1!

symmetry

(b) $P(E) = P(E|s_1)$

$= P(r_2 \gtrless 2r_1 | s_1)$

$= P(n_1 + n_2 \gtrless 2\sqrt{E_s} + 2n_1)$

$= P(n_2 - n_1 \gtrless 2\sqrt{E_s})$

Let $n = n_2 - n_1$

$n \sim N(0, N_0)$

Thus, $P(\epsilon) = P\left(n > 2\sqrt{E_s}\right)$

$$= \int_{2\sqrt{E_s}}^{\infty} \frac{1}{\sqrt{\pi 2 N_0}} e^{-x^2/2N_0}\, dx$$

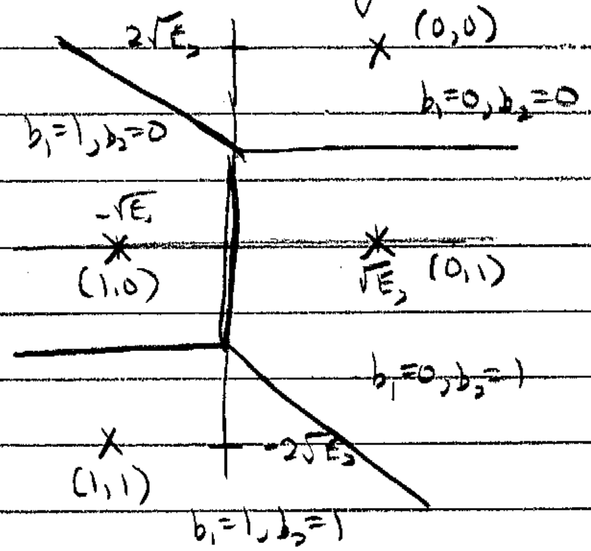
$$= Q\left(2\sqrt{\frac{E_s}{N_0}}\right)$$

(c)

$$P_{discard}(\epsilon) = Q\left(\sqrt{\frac{2E_s}{N_0}}\right) \qquad \text{(BPSK!)}$$

3 dB more energy required.

6) (a) Just like a 4-ary signal set:



(b) Union Bound

d_{ij}

i \ j	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	0	$2\sqrt{E_s}$	$2\sqrt{2}\sqrt{E_s}$	$\sqrt{20}\sqrt{E_s}$
(0,1)	$2\sqrt{E_s}$	0	$2\sqrt{E_s}$	$2\sqrt{2}E_s$
(1,0)	$2\sqrt{2}\sqrt{E_s}$	$2\sqrt{E_s}$	0	$2\sqrt{E_s}$
(1,1)	$\sqrt{20}\sqrt{E_s}$	$2\sqrt{2}\sqrt{E_s}$	$2\sqrt{E_s}$	0

$$P(E) \le \frac{1}{m}\sum_{i=1}^{m}\sum_{j\neq i} Q\left(\frac{d_{ij}}{\sqrt{2N_0}}\right)$$

$$= \frac{1}{4}\left(6\,Q\left(\sqrt{\frac{2E_s}{N_0}}\right) + 4\,Q\left(\sqrt{\frac{4E_s}{N_0}}\right) + 2\,Q\left(\sqrt{\frac{10E_s}{N_0}}\right)\right.$$

tight at high SNR

(c) Choose $b_1 = 0$ if

$$p(\underline{r} \mid b_1 = 0) \geq p(\underline{r} \mid b_1 = 1)$$

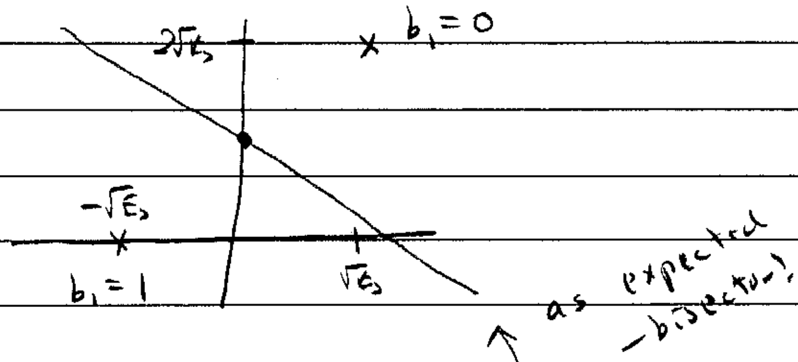
Use Law of Total Probability:

$$\sum_{i \in \{0,1\}} p(\underline{r} \mid b_1 = 0, b_0 = i)\, \cancelto{1/2}{p(b_0 = i)} \geq p(\underline{r} \mid b_1 = 1, b_0 = i)\, \cancelto{1/2}{p(b_0 = i)}$$

$$\frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r_1 - \sqrt{E_s})^2 + (r_2 - 2\sqrt{E_s})^2}{N_0}} + \frac{1}{\sqrt{\pi N_0}} e^{-\frac{(r_1^2 + (r_2 - \sqrt{E_s})^2)}{N_0}}$$

$$\geq \frac{1}{\sqrt{\pi N_0}} e^{-\frac{((r_1 + \sqrt{E_s})^2 + r_2^2)}{N_0}} + \frac{1}{\sqrt{\pi N_0}} e^{\frac{(-(r_1 + \sqrt{E_s})^2 + (r_2 + 2\sqrt{E_s})^2)}{N_0}}$$

(d) Suppose $b_2 = 0$. Signal set is:



min Euclidean distances choose $b_1 = 0$ if

$$(r_1 - \sqrt{E_s})^2 + (r_2 - 2\sqrt{E_s})^2 \geq (r_1 + \sqrt{E_s})^2 + r_2^2$$

$$\cancel{r_1^2} - 2 r_1 \sqrt{E_s} + \cancel{E_s} + \cancel{r_2^2} - 4\sqrt{E_s}\, r_2 + 4 E_s \geq \cancel{r_1^2} + 2 r_1 \sqrt{E_s} + \cancel{E_s} + \cancel{r_2^2}$$

$$\cancelto{\sqrt{E_s}}{4 E_s} \geq \cancel{4} r_1 \cancel{\sqrt{E_s}} + \cancel{4} \cancel{\sqrt{E_s}}\, r_2$$

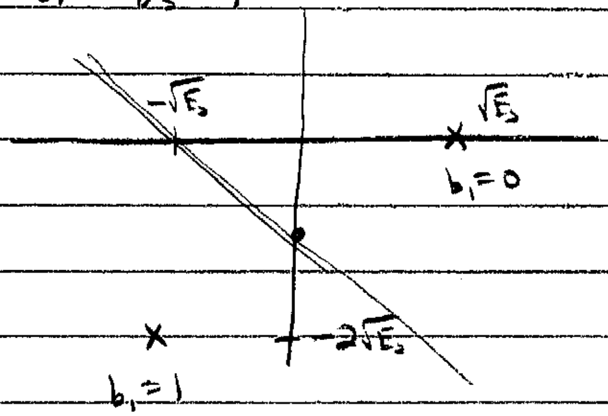
$$r_2 \leq \sqrt{E_s} - r_1$$

If $b_2 = 0$

$$P(\hat{b}_1 \neq b_1) = Q\left(d / \sqrt{2 N_0}\right)$$

$$= Q\left(\sqrt{\frac{4 E_s}{N_0}}\right)$$

Likewise, for $b_2 = 1$



Choose $b_1 = 0$ if $r_2 \geq -\sqrt{E_s} - r_1$

$$P(\hat{b}_1 \neq b_1) = Q\left(d/\sqrt{2N_0}\right) = Q\left(\sqrt{\frac{4E_s}{N_0}}\right)$$

Overall,

$$P(\hat{b}_1 \neq b_1) = P(\hat{b}_1 \neq b_1 \mid b_2 = 0)P(b_2 = 0) + P(\hat{b}_1 \neq b_1 \mid b_2 = 1)P(b_2 = 1)$$
$$= Q\left(\sqrt{\frac{4E_s}{N_0}}\right)$$

(e) $n_1 \sim N(0, N_0/2)$

$$E[\tilde{n}] = E[s_2] + E[n_2] = 0$$
$$E[\tilde{n}^2] = E[s_2^2] + E[n_2^2] = E_s + N_0/2$$
$$\Rightarrow \tilde{n} \sim N(0, E_s + N_0/2)$$

Choose $b_1 = 1$ if

$$p(\underline{r} \mid b_1 = 1) \geq p(\underline{r} \mid b_1 = 0)$$

$$\cancel{\frac{1}{\sqrt{\pi N_0}}} e^{-(r_1+\sqrt{E_s})^2/N_0} \cancel{\frac{1}{\sqrt{2\pi(E_s+N_0/2)}}} e^{-(r_2+\sqrt{E_s})^2/2(E_s+N_0/2)}$$

$$\geq \cancel{\frac{1}{\sqrt{\pi N_0}}} e^{-(r_1-\sqrt{E_s})^2/N_0} \cancel{\frac{1}{\sqrt{2\pi(E_s+N_0/2)}}} e^{-(r_2-\sqrt{E_s})^2/2(E_s+N_0/2)}$$

$$+(r_1+\sqrt{E_s})^2/N_0 + (r_2+\sqrt{E_s})^2/2(E_s+N_0/2)$$

$$\leq +(r_1-\sqrt{E_s})^2/N_0 + (r_2-\sqrt{E_s})^2/2(E_s+N_0/2)$$

$$\frac{\cancel{r_1^2} + 2r_1\sqrt{E_s} + \cancel{E_s}}{N_0} + \frac{\cancel{r_2^2} + 2r_2\sqrt{E_s} + \cancel{E_s}}{2(E_s+N_0/2)}$$

$$\leq \frac{\cancel{r_1^2} - 2r_1\sqrt{E_s} + \cancel{E_s}}{N_0} + \frac{\cancel{r_2^2} - 2r_2\sqrt{E_s} + \cancel{E_s}}{2(E_s+N_0/2)}$$

$$\frac{\cancel{4}r_1\cancel{\sqrt{E_s}}}{N_0} \leq \frac{-\cancel{4}r_2\cancel{\sqrt{E_s}}}{2(E_s+N_0/2)}$$

$$r_2 \leq -2r_1\left(E_s/N_0 + 1/2\right)$$