1. The cumulative distribution function of a random variable $X$ is given as

$$F_X(x) = \begin{cases} 
0, & \text{for } x < 0 \\
\frac{2}{3}, & \text{for } 0 \leq x < 1 \\
\frac{(x+1)}{3}, & \text{for } 1 \leq x < 2 \\
1, & \text{for } 2 \leq x 
\end{cases}.$$

(a) Find the following probabilities:

- $P(X < 1)$.
- $P(1 \leq X \leq 1.5)$.
- $P(X = 1)$.

(b) Find and sketch the probability density function of $X$.

(c) Find $E[X]$ and $\text{Var}[X]$.

2. Suppose random variable $X$ has probability density function

$$f_X(x) = 0.5 \delta(x) + c x^2 [u(x+1) - u(x-1)].$$

where the function $u(x)$ is defined as:

$$u(x) = \begin{cases} 
1, & x \geq 0 \\
0, & \text{else} 
\end{cases}.$$

(a) Find $c$.

(b) Find the cumulative distribution function of $X$.

(c) Find the following probabilities:

- $P(X = 0)$
- $P(X = 1)$
- $P(-0.5 < X \leq 0)$
- $P(-1 < X < 0)$
3. The probability density function of a random variable $X$ is given by $f_X(x)$, where:

$$f_X(x) = \begin{cases} 
  cx^2, & -2 \leq x \leq 1 \\
  0, & \text{otherwise}
\end{cases}$$

(a) Find the value of the constant $c$.

(b) Find the probability that $X^2 \geq 1$.

(c) Find the probability that $X - 1 \geq -\frac{1}{4}$.

(d) Let the random variable $Y$ be defined by:

$$Y = \begin{cases} 
  -X, & X \leq 0 \\
  0, & X \geq 0
\end{cases}$$

Find the probability density function of $Y$.

4. Suppose that I am observing a network connection that is good (“G”) with probability 0.9 and bad (“B”) with probability 0.1. Let $T$ (in seconds) be the time until the first packet arrives. If the connection is good, $T$ has probability density function:

$$f_G(x) = \begin{cases} 
  2e^{-2x}, & x \geq 0 \\
  0, & \text{else}
\end{cases}$$

If the connection is bad, $T$ has probability density function:

$$f_B(x) = \begin{cases} 
  e^{-x}, & x \geq 0 \\
  0, & \text{else}
\end{cases}$$

(a) What is the probability that the first packet arrives during the first second?

(b) Given that the first packet arrives during the first second, what is the probability that the link is good?

5. (a) Suppose a stick of length $L$ is broken into two pieces at a random point along its length, all points equally likely. Let the random variable $X$ represent the length of the shorter one of the two pieces. Find the cumulative distribution function $F_X(x)$ and probability density function $f_X(x)$ for $X$.

(b) Repeat part (a), if the likelihood of a breaking point is proportional to the square of the distance between the point and the closer of the two endpoints.

(c) Repeat part (a), but now find the cumulative distribution function $F_Y(y)$ and probability density function $f_Y(y)$ for the random variable $Y$ if all break points are equally likely (as in (a)), but with $Y$ defined as the difference of the lengths of the long and the short pieces.
6. A service facility charges a $20 fixed fee plus $25 per hour of service up to 6 hours, and no additional fee is charged for service exceeding 6 hours. Suppose that the service time $\tau$ is equally likely to be any time in $[0, 10]$ hours (Note that $\tau$ is a continuous random variable). Let $X$ represent the cost of service in the facility.

(a) Find and sketch the cumulative distribution function for $X$.

(b) Find and sketch the probability density function for $X$.

(c) What is the probability that you end up paying less than $60 for service? Answer this part two different ways:

- Using your answer to part (b).
- Finding the time (call it $\tau_0$) at which the service would cost exactly $60 and then finding the probability that $\tau \leq \tau_0$.

7. Use linearity of expectation to show that $\text{Var}(aX) = a^2 \text{Var}(X)$, where $a$ is a constant.

8. Let $X$ be a Gaussian random variable with mean $\mu = 3$ and variance $\sigma^2 = 1$. (Note: Many of your answers will be in terms of the erf($\cdot$) function.)

(a) Sketch $f_X(x)$, the probability density function of $X$.

(b) Find the probability that $-4 \leq X \leq 1$.

(c) Find the probability that $X^2 \geq 10$.

(d) Find $F_X(x|A)$, that is, find the cumulative distribution function of $X$ given that the event $A$ has occurred, where $A = \{X \geq 4\}$.

9. You have a table that gives you the value of the “Goeckel Function” for all $x \geq 0$:

$$G(x) = \int_x^\infty \frac{1}{2} e^{-\frac{u^2}{2}} du$$

Let $Y$ be a Gaussian random variable with mean $\mu$ and variance $\sigma^2$; that is, $Y \sim N(\mu, \sigma^2)$. Write an expression for $P(Y \leq y)$ for all $y$ in terms of $G(x)$. 