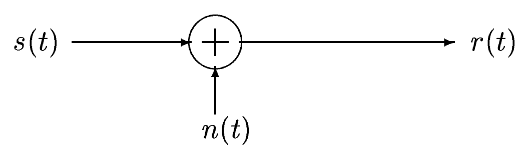
ECE 564/645 - Digital Communications, Spring 2014
Homework #3
Due: March 7, 2014 (in class)

1. Consider the following on-off keying system for transmitting a bit b_0, **equally likely** to be 0 or 1, in $t \in (0,1)$. For $b_0 = 0$, we let $s(t) = 0$, and for $b_0 = 1$, we let $s(t) = \sqrt{2E_s}p(t)$, where

$$p(t) = \begin{cases} 1, & 0 \leq t \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

The signal $s(t)$ is transmitted across a channel modeled as the following:



where $n(t)$ is additive white Gaussian noise with power spectral density $\frac{N_0}{2}$ and $r(t)$ is the received waveform.

(a) Find the receiver for processing $r(t), t \in (0,1)$ to obtain an estimate for the transmitted bit that minimizes the probability of a bit error.

(b) Find the probability of a bit error in terms of the average energy per symbol E_s and N_0.

(c) How much better or worse (in dB of $\frac{E_s}{N_0}$) is this system than a binary phase-shift keyed (BPSK) system operating on an AWGN channel?

2. Consider the 2-dimensional vector channel $\underline{r} = \underline{s} + \underline{n}$ where $\underline{r}$ is the received vector, $\underline{s} = \underline{s}_i$ if message m_i is to be sent on $(0, T_s)$, and $\underline{n} = (n_1, n_2)^T$. Let n_1 and n_2 be independent Gaussian random variables with mean 0 and variance $\frac{N_0}{2}$ (i.e. this is an AWGN vector channel). Suppose there are four possible messages ($M = 4$) and

$$\underline{s}_1 = (0,0)^T$$

$$\underline{s}_2 = \left(\frac{d}{\sqrt{2}}, \frac{d}{\sqrt{2}}\right)^T$$

$$\underline{s}_3 = \left(0, \sqrt{2}d\right)^T$$

$$\underline{s}_4 = \left(-\frac{d}{\sqrt{2}}, \frac{d}{\sqrt{2}}\right)^T$$

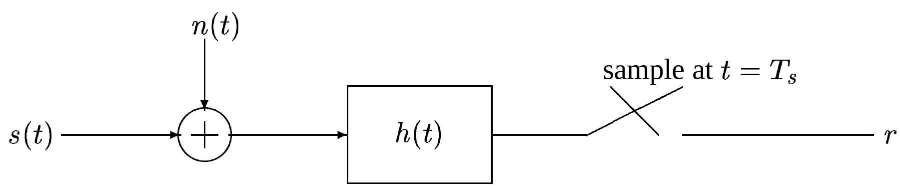
(a) Assuming equally likely messages (i.e. $p(m_i) = \frac{1}{4}, \forall i$) and MAP reception, draw the decision regions in $\underline{r}$-space ($\underline{r} = (r_1, r_2)^T$), showing where each signal is chosen. You do not have to give precise intercepts and such for the boundaries - just approximate the picture.

(b) For equally likely messages and the MAP receiver of (a):

- Find the symbol error probability $P(E)$ in terms of d and N_0.
- Find the symbol error probability $P(E)$ in terms of N_0 and the average symbol energy E_s.

(c) How much better (or worse) is this signal set (in dB) than QPSK?

3. Consider the following communication system:



where $n(t)$ is white Gaussian noise with power spectral density $\frac{N_0}{2}$, and $h(t)$ is a linear time-invariant filter applied to $r(t)$. Note that r is a scalar. For $t \in (0, T_s)$, we wish to communicate one of two **equally likely** messages m_1 and m_2 with respective signals

$$s_1(t) = 0$$

$$s_2(t) = \begin{cases} \sqrt{\frac{1}{T_s}}, & t \in (0, T_s) \\ 0, & \text{otherwise} \end{cases}$$

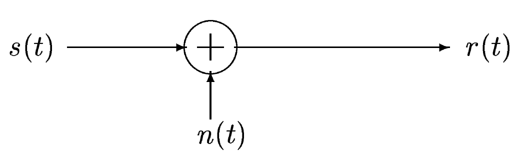
(a) Find the optimal filter $h(t)$ and processing of r to implement the MAP receiver on $r(t)$ to decide between m_1 and m_2.

(b) Suppose instead of our filter from (a) that the factory ships us the filter

$$h(t) = \begin{cases} -\frac{t}{T_s}, & t \in (0, T_s) \\ 0, & \text{otherwise} \end{cases}$$

Given that we employ this filter in our system, what is the processing of r that minimizes the probability that the wrong message is chosen. Be sure to justify your answer.

4. Consider the waveform channel:



where $n(t)$ is additive white Gaussian noise with power spectral density $\frac{N_0}{2}$, $r(t)$ is the received waveform, and $s(t) = s_i(t)$ when message m_i is to be sent during time $t \in (0, T_s)$. Suppose there are $M = 2$ possible **equally likely** messages and the corresponding signals are:

$$\begin{aligned} s_1(t) &= \sqrt{2P_c}\cos(2\pi f_c t), \quad 0 \le t \le T_s \\ s_2(t) &= -\sqrt{2P_c}\cos(2\pi f_c t), \quad 0 \le t \le T_s \end{aligned}$$

where P_c is the transmitted power and f_c is the carrier frequency.

(a) Specify the optimal (MAP) processing of $r(t), t \in [0, T_s]$ for determining which message was sent. What is the probability of error of the MAP receiver in terms of $E_s = P_c T_s$ and N_0?

(b) The **same exact processing from (a)** is employed but **unknown to the receiver,** the transmitted signals are really given by

$$\begin{aligned} s_1(t) &= \sqrt{2P_c}\cos(2\pi f_c t + \theta_\epsilon), \quad 0 \le t \le T_s \\ s_2(t) &= -\sqrt{2P_c}\cos(2\pi f_c t + \theta_\epsilon), \quad 0 \le t \le T_s \end{aligned}$$

where $0 \le \theta_\epsilon \le \frac{\pi}{2}$. Find the probability of error of this system as a function of θ_ϵ. (Assume $f_c \gg \frac{1}{T_s}$).

5. Consider the following channel:

$$r_1 = s + n_1 \qquad\qquad r_2 = n_1 + n_2$$

where r_1 and r_2 are observable at the receiver, and n_1 and n_2 are independent zero-mean Gaussian random variables with variance $\frac{N_0}{2}$. The channel is used to transmit one of two equally likely messages m_1 and m_2, where $s = s_1 = \sqrt{E_s}$ if m_1 is to be sent, and $s = s_2 = -\sqrt{E_s}$ if m_2 is to be sent.

(a) Find the MAP decision rule. Explain why it depends on r_2, even though r_2 "is only noise."

(b) Find the probability of error of the MAP receiver.

(c) Suppose we discarded signal r_2 because "it is only noise." How much more energy (in dB) would we have to use with this suboptimal system as compared to the system of (a) to achieve the same probability of error?

6. [645 only] Two users each want to transmit one information bit across a pair of channels. User 1 uses both channels, while User 2 employs only the second channel so that:

$$\begin{aligned} r_1 &= s_1 + n_1 \\ r_2 &= s_1 + s_2 + n_2 \end{aligned}$$

where the channel noises n_1 and n_2 are independent Gaussian random variables with mean 0 and variance $\frac{N_0}{2}$.

The user signals are given by:

$$s_1 = \begin{cases} \sqrt{E_s}, & b_1 = 0 \\ -\sqrt{E_s}, & b_1 = 1 \end{cases} \quad \text{and} \quad s_2 = \begin{cases} \sqrt{E_s}, & b_2 = 0 \\ -\sqrt{E_s}, & b_2 = 1 \end{cases}.$$

The information bits are independent of each other, and the channel noises and information bits are mutually independent. Let the receiver bit estimates for User 1 and User 2 be given by $\hat{b}_1$ and $\hat{b}_2$, respectively.

(a) Find the receiver (with inputs r_1, r_2 and outputs $\hat{b}_1$, $\hat{b}_2$) that minimizes the probability that $(b_1, b_2) \neq (\hat{b}_1, \hat{b}_2)$. (In other words, it maximizes the probability that **both** bits are correctly detected). Sketch the decision regions in terms of r_1 and r_2.

(b) Let the probability of error $P(E)$ of the receiver in (a) be the probability that $(b_1, b_2) \neq (\hat{b}_1, \hat{b}_2)$. Find an approximation $\tilde{P}(E)$ to this error probability such that:

$$\lim_{\frac{E_s}{N_0} \to \infty} \frac{\tilde{P}(E)}{P(E)} = 1$$

(c) Find the equation(s) that can be solved to determine the receiver (with inputs r_1, r_2 and output $\hat{b}_1$) that minimizes the probability that $b_1 \neq \hat{b}_1$. There is no need to simplify the equation(s), **but you must give each quantity in the equation(s) explicitly**.

(d) Repeat part (c), but assume that b_2 is also known at the receiver. In this case, solve the equations for the optimal receiver, and find the probability that $b_1 \neq \hat{b}_1$ for this receiver.

(e) Suppose that a receiver for User 1 treats the quantity $\tilde{n} = s_2 + n_2$ as a Gaussian random variable; in other words, the receiver for User 1 assumes that $\tilde{n}$ is Gaussian with the same mean and variance as $s_2 + n_2$. Find the receiver (with inputs r_1, r_2 and output $\hat{b}_1$) that minimizes the probability that $b_1 \neq \hat{b}_1$ under this assumption. Sketch the decision regions in terms of r_1 and r_2.