1. Suppose we run a real bandpass signal $x(t)$, whose Fourier transform is nonzero only for $f_c - W \leq |f| \leq f_c + W$ (assume $W \ll f_c$), through a Hilbert Transformer $h(t)$, whose frequency response is given by:

$$H(f) = \begin{cases} -j & f \geq 0 \\ j & f < 0 \end{cases}$$

to arrive at $\hat{x}(t) = x(t) * h(t)$, which is defined as the *Hilbert Transform* of $x(t)$.

(a) Show that the Hilbert Transform of $x(t) = \pi(t) \cos(2\pi f_c t) - x_Q(t) \sin(2\pi f_c t)$ is given by $\hat{x}(t) = x_I(t) \sin(2\pi f_c t) + x_Q(t) \cos(2\pi f_c t)$. (*Hint: Do this in the frequency domain.*)

(b) Define the signal $z(t) = x(t) + j\hat{x}(t)$. Using part (a), show that the complex envelope $x_z(t)$, as defined in class, is given by: $x_z(t) = z(t)e^{-j2\pi f_c t}$. (*Hint: Do this in the time domain.*)

2. The bandpass signal $x(t) = \text{sinc}(2\pi f_c t)$ is passed through a bandpass filter with impulse response $h(t) = \text{sinc}^2(2\pi f_c t)$ to yield the output $y(t)$.

(a) Find the complex envelopes of $x(t)$ and $h(t)$ (i.e. find $x_z(t)$ and $h_z(t)$, respectively).

(b) Find and sketch the Fourier transforms of $x_z(t)$ and $h_z(t)$.

(c) Find $y_z(t)$, the complex envelope of $y(t)$, and use it to find $y(t)$.

3. Suppose that I have a signal $x(t)$ with Fourier transform

$$X(f) = \frac{1}{400} \left( \frac{f - f_c}{200} \right) + \frac{1}{400} \left( \frac{f + f_c}{200} \right)$$

where $f_c \gg 1000$.

(a) Find real lowpass signals $x_I(t)$ and $x_Q(t)$ such that $x(t) = x_I(t) \cos(2\pi f_c t) - x_Q(t) \sin(2\pi f_c t)$

(b) Now suppose that I filter the signal $x(t)$ with a filter with impulse response $h(t)$, where the Fourier transform of $h(t)$ is given by:

$$H(f) = \begin{cases} 1, & |f| \leq f_c \\ 0, & |f| > f_c \end{cases}$$

Let the output of the filter be denoted $y(t) = h(t) * x(t)$. Find real lowpass signals $y_I(t)$ and $y_Q(t)$ such that:

$$y(t) = y_I(t) \cos(2\pi f_c t) - y_Q(t) \sin(2\pi f_c t)$$
4. In your first job after graduation from UMass, you are assigned to design a transmitter that outputs
\[ y(t) = h(t) * x(t), \]
where
\[ x(t) = x_I(t) \cos(2\pi 10^6 t) - x_Q(t) \sin(2\pi 10^6 t) \]
is a bandpass signal with lowpass signals \( x_I(t) \) and \( x_Q(t) \) (each of bandwidth 5 KHz) as its in-phase and quadrature components, respectively, and \( h(t) \) is a real bandpass filter. The inputs to your transmitter are \( x_I(t) \) and \( x_Q(t) \), and the output is \( y(t) \).

(a) Suppose that the bandpass filter response is specified by \( H(f) \), the Fourier transform of \( h(t) \):

\[
\begin{array}{c}
10^6 \\
\hline \hline
3j \\
\hline \hline
-3j
\end{array}
\]

Sketch the Fourier transforms \( H_I(f) \) and \( H_Q(f) \) of the in-phase part \( h_I(t) \) and quadrature part \( h_Q(t) \), respectively, of the filter \( h(t) \). Hint: Recall from class that
\[
H_I(f) = \frac{H_Z(f) + H_Z^*(-f)}{2},
\]
\[
H_Q(f) = \frac{H_Z(f) - H_Z^*(-f)}{2j},
\]
where \( H_Z(f) \) is the Fourier transform of the complex envelope of \( h(t) \).

(b) Draw a circuit that takes as input \( x_I(t) \) and \( x_Q(t) \) and outputs \( y(t) \), while employing only summers, multipliers, oscillators, and lowpass filters. (Note: A “lowpass filter” is defined for this problem as one whose frequency response is non-zero only for \( |f| \leq 5\text{kHz} \).)

(c) Find the output \( y(t) \) of your transmitter when \( x(t) = 6 \cos(2\pi 1,002,500t + \frac{\pi}{2}) \).

(d) Suppose that \( X(f) = H(f) \). Determine whether \( x(t) \), the inverse Fourier transform of \( X(f) \), could be the output of a DSB-SC system; that is, of the form
\[
x(t) = A_m(t) \cos(2\pi f_c t + \theta)
\]