Homework #2 Solutions

ECE 564/645 Spring 2014

1) (a) $P_X = R_X(0) = 1$

(b) $X(t)$ is a Gaussian RP $\Rightarrow$ $X(3)$ is Gaussian

$$E[X(3)] = 0 \quad Var[X(3)] = E[X^2(3)] - (\cancel{E[X(3)]})^{2}{}^{0}$$
$$= 1$$

$$P(X(3) > 1) = \int_1^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = Q(1)$$

(c)
$$S_X(f) = \mathcal{F}\{R_X(\tau)\} = \int_{-\infty}^{\infty} e^{-|\tau|/2} e^{-j2\pi f\tau} d\tau$$
$$= \int_{-\infty}^{0} e^{\tau/2} e^{-j2\pi f\tau} d\tau + \int_{0}^{\infty} e^{-\tau/2} e^{-j2\pi f\tau} d\tau$$
$$= \int_{-\infty}^{0} e^{(\tau/2 - j2\pi f\tau)} d\tau + \int_{0}^{\infty} e^{(-\tau/2 - j2\pi f\tau)} d\tau$$
$$= \frac{1}{1/2 - j2\pi f} + \frac{1}{1/2 + j2\pi f}$$
$$= \frac{1}{1/4 + 4\pi^2 f^2}$$

(d) $$S_X(f) = \frac{1}{1/4 + 4\pi^2 f^2} = |H(f)|^2 \cdot N_0/2$$

$$\Rightarrow |H(f)| = \sqrt{\frac{2}{N_0}} \frac{1}{1/2 - j2\pi f}$$

(e) $X(t)$ is Gaussian $\Rightarrow X(0), X(1), X(2)$ jointly Gaussian

$\Rightarrow X(0)+X(1)+X(2)$ is Gaussian

$$E[Z] = E[X(0)] + E[X(1)] + E[X(2)] = 0$$

$$E[Z^2] = E[(X(0)+X(1)+X(2))^2]$$

$$= E[\overset{1}{\cancel{X^2(0)}} + \overset{1}{\cancel{X^2(1)}} + \overset{1}{\cancel{X^2(2)}} + 2\overset{R_X(1)}{\cancel{X(1)X(2)}} + 2\overset{R_X(2)}{\cancel{X(1)X(3)}} + 2\overset{R_X(1)}{\cancel{X(2)X(3)}}]$$

$$= 3 + 4e^{-1/2} + 2e^{-1}$$

$$\sigma_Z^2 = E[Z^2] - (E[Z])^2 \quad \text{and} \quad f_Z(z) = \frac{1}{\sqrt{2\pi\sigma_Z^2}} e^{-z^2/2\sigma_Z^2}$$

(f) Following (e), $X(0)+X(2) \sim N(0, 2+2e^{-1})$

$$P(X(0)+X(2) > 3) = Q\left(\frac{3}{\sqrt{2+2e^{-1}}}\right)$$

(g)

$$E[Y] = E\left[\int_0^{T_s} N(t)\,dt\right] = \int_0^{T_s} \overset{0}{\cancel{E[N(t)]}}\,dt = 0$$

$$E[Y^2] = E\left[\int_0^{T_s} N(t)\,dt \int_0^{T_s} N(s)\,ds\right]$$

$$= \int_0^{T_s}\int_0^{T_s} \overset{N_0/2\ \delta(t-s)}{\cancel{E[N(t)N(s)]}}\,dt\,ds$$

$$= \int_0^{T_s} N_0/2\,dt = N_0T_s/2$$

$$E[Y^2] - (E[Y])^2 = N_0T_s/2$$

(h) $Y \sim N(0, N_0T_s/2)$

$$P(Y>2) = Q\left(2\Big/\sqrt{\frac{N_0T_s}{2}}\right)$$

2)

(a)

$$\int_0^{T_s} s_m(t)\, s_n(t)\, dt$$

$$= \int_0^{T_s} \sum_{j=1}^{N} s_{m,j}\, \phi_j(t) \sum_{k=1}^{N} s_{n,k}\, \phi_k(t)\, dt$$

$$= \sum_{j=1}^{N} \sum_{k=1}^{N} s_{m,j}\, s_{n,k} \cancelto{\delta_{j,k}}{\int_0^{T_s} \phi_j(t)\, \phi_k(t)\, dt}$$

$$= \sum_{j=1}^{N} s_{m,j}\, s_{m,j}$$

(b)

$$\int_0^{T_s} \left(s_m(t) - s_n(t)\right)^2 dt$$

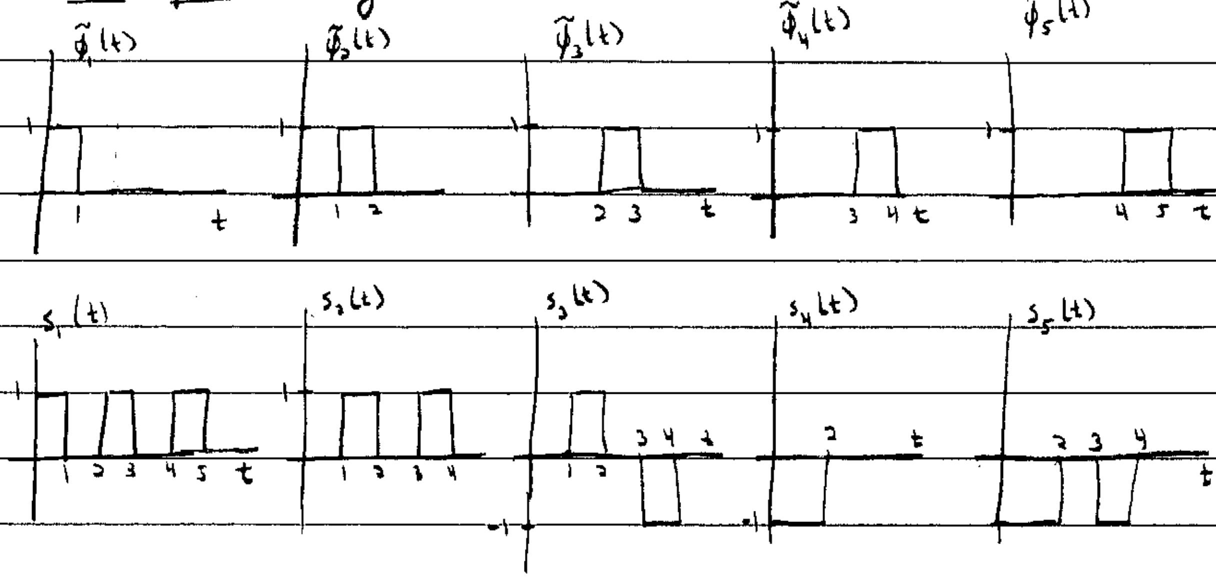
$$= \underbrace{\int_0^{T_s} s_m^2(t)\, dt}_{\sum_{j=1}^{N} s_{m,j}^2} - 2 \underbrace{\int_0^{T_s} s_m(t)\, s_n(t)\, dt}_{\sum_{j=1}^{N} s_{m,j}\, s_{n,j}} + \underbrace{\int_0^{T_s} s_n^2(t)\, dt}_{\sum_{j=1}^{N} s_{n,j}^2}$$

$$= \sum_{j=1}^{N} \left(s_{m,j}^2 - 2 s_{m,j}\, s_{n,j} + s_{n,j}^2\right)$$

$$= \sum_{j=1}^{N} \left(s_{m,j} - s_{n,j}\right)^2$$

3) (a)

<u>One possibility</u>:



(b) No. It does not.

I will do G-S on the <u>vectors</u>, but the waveform dot products, energies will all be the same.

$$\underline{\phi}_1 = \frac{\underline{s}_1}{\|\underline{s}_1\|} = \left(1/\sqrt{3},\ 0,\ 1/\sqrt{3},\ 0,\ 1/\sqrt{3} \right)^T$$

$$\underline{v}_2 = \underline{s}_2 - \left(\cancel{\underline{s}_2^T \underline{\phi}_1}\right)^{0} \underline{\phi}_1 = \underline{s}_2$$

$$\Rightarrow \phi_2 = \frac{\underline{v}_2}{\|\underline{v}_2\|} = \left(0,\ 1/\sqrt{2},\ 0,\ 1/\sqrt{2},\ 0\right)^T$$

$$\underline{v}_3 = \underline{s}_3 - \left(\cancel{\underline{s}_3^T \underline{\phi}_1}\right)^{0} \underline{\phi}_1 - \left(\cancel{\underline{s}_3^T \underline{\phi}_2}\right)^{0} \phi_2$$

$$= (0, 1, 0, -1, 0)^T \Rightarrow \phi_3 = \frac{\underline{v}_3}{\|\underline{v}_3\|} = \left(0, 1/\sqrt{2}, 0, -1/\sqrt{2}, 0\right)^T$$

$$\underline{v}_4 = \underline{s}_4 - \overset{-1/\sqrt{3}}{(\underline{s}_4^T \underline{\phi}_1)}\,\underline{\phi}_1 - \overset{-1/\sqrt{2}}{(\underline{s}_4^T \underline{\phi}_2)}\,\underline{\phi}_2 - \overset{-1/\sqrt{2}}{(\underline{s}_4^T \underline{\phi}_3)}\,\underline{\phi}_3$$

$$= (-1,-1,0,0,0)^T + (1/3,0,1/3,0,1/3)^T + (0,1/2,0,1/2,0)^T + (0,-1/2,0,-1/2,0)^T$$

$$= (-2/3,\ 0,\ 1/3,\ 0,\ 1/3)^T$$

$$\underline{\phi}_4 = \frac{v_4}{\|v_4\|} = \left(-\sqrt{2/3},\ 0,\ 1/\sqrt{6},\ 0,\ 1/\sqrt{6}\right)^T$$

$$\underline{v}_5 = \underline{s}_5 - \overset{-1/\sqrt{3}}{(\underline{s}_5^T \underline{\phi}_1)}\,\underline{\phi}_1 - \overset{-\sqrt{2}}{(\underline{s}_5^T \underline{\phi}_2)}\,\underline{\phi}_2 - \overset{0}{(\underline{s}_5^T \underline{\phi}_3)}\,\underline{\phi}_3 - \overset{\sqrt{2}/\sqrt{3}}{(\underline{s}_5^T \underline{\phi}_4)}\,\underline{\phi}_4$$

$$= (-1,-1,0,-1,0)^T + (1/3,0,1/3,0,1/3)^T + (0,1,0,1,0)^T + (2/3,0,-1/3,0,-1/3)^T$$

$$= 0$$

$\Rightarrow$ 4-dimensional space

$\underline{s}_1 = (\sqrt{3}, 0, 0, 0)$

$\underline{s}_2 = (0, \sqrt{2}, 0, 0)$ $\qquad \tilde{\underline{s}}_i^T \phi_j$

$\underline{s}_3 = (0, 0, \sqrt{2}, 0)$

$\underline{s}_4 = (-1/\sqrt{3}, -1/\sqrt{2}, -1/\sqrt{2}, \sqrt{2/3})$

$\underline{s}_5 = (-1/\sqrt{3}, -\sqrt{2}, 0, \sqrt{2/3})$

4)

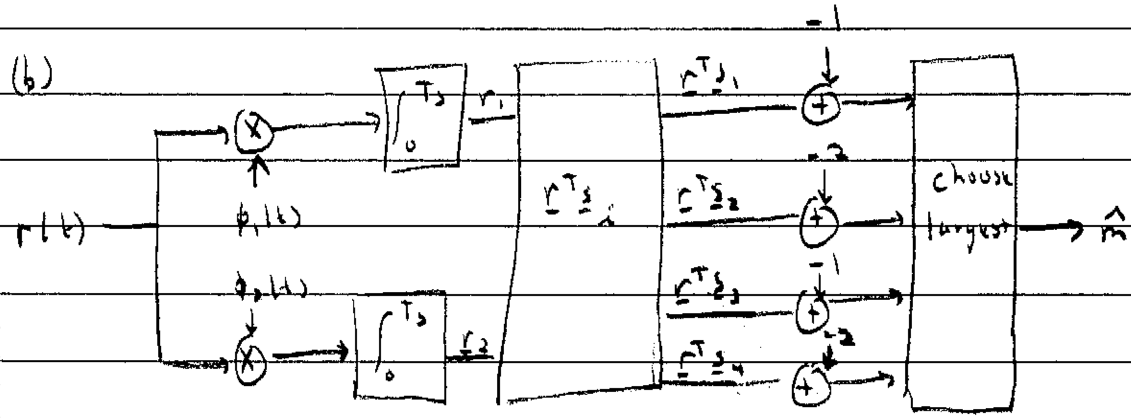
(a) $s_1(t) = \sqrt{2/T_s}\,\sin(2\pi f_c t)$ $\qquad \underline{s}_1 = (1, 0)$

$s_2(t) = \sqrt{2/T_s}\,\cos(2\pi f_c t) - \sqrt{2/T_s}\,\sin(2\pi f_c t)$ $\qquad \underline{s}_2 = (-1, 1)$

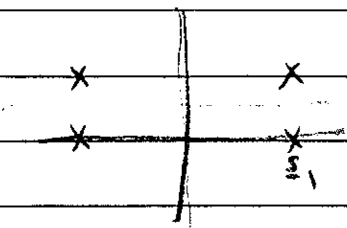
$s_3(t) = -\sqrt{2/T_s}\,\sin(2\pi f_c t)$ $\qquad \underline{s}_3 = (-1, 0)$

$s_4(t) = \sqrt{2/T_s}\,\cos(2\pi f_c t) + \sqrt{2/T_s}\,\sin(2\pi f_c t)$ $\qquad \underline{s}_4 = (1, 1)$

(b)



(c)

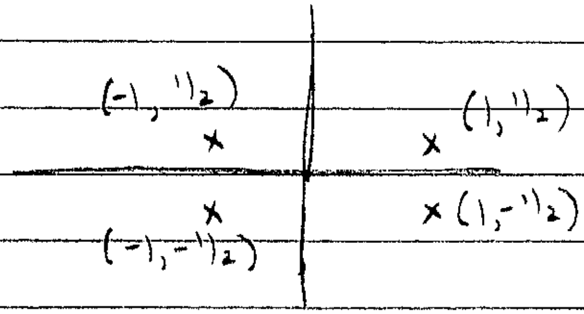


$$P(E) = 1 - P(c)$$
$$= 1 - P(c \mid \underline{s}_1)$$
$$= 1 - P(\{n_1 > 1\} \cap \{n_2 < 1/2\})$$
$$= 1 - (1 - P(n_1 \le 1)) \cdot (1 - P(n_2 > 1/2))$$
$$= Q\left(\sqrt{2/N_0}\right) + Q\left(\sqrt{1/N_0}\right) - 2\,Q\left(\sqrt{2/N_0}\right) Q\left(\sqrt{1/N_0}\right)$$

$$E_s = \frac{1}{4}\cdot 1 + \tfrac{1}{4}\cdot 2 + \tfrac{1}{4}\cdot 1 + \tfrac{1}{4}\cdot 2 = 3/2$$

$$\Rightarrow P(E) = Q\left(\sqrt{\frac{4}{3}\, E_s/N_0}\right) + Q\left(\sqrt{\frac{2}{3}\frac{E_s}{N_0}}\right) - Q\left(\sqrt{\frac{4}{3}\frac{E_s}{N_0}}\right) Q\left(\sqrt{\frac{2}{3}\frac{E_s}{N_0}}\right)$$

(d)



Now, $E_s = 1^2 + (1/2)^2 = 5/4$ (and same distances)

energy savings: $\frac{5/4}{3/2} = 5/6 = 0.79$ dB