1. Let $X(t)$ be a wide-sense stationary Gaussian random process with mean zero and autocorrelation $R_X(\tau) = e^{-|\tau|/\sigma^2}$. Let $N(t)$ be a white Gaussian noise process with power spectral density $\frac{N_0}{2}$.

(a) Find $P_x$, the power in $X(t)$.

(b) Find $P(X(3) > 1)$.

(c) Find the power spectral density $S_X(f)$ of $X(t)$.

(d) Find a filter (give $h(t)$ or $H(f)$) that has input $N(t)$ and output with power spectral density $S_X(f)$.

(e) Let $Z = X(0) + X(1) + X(2)$. Find $f_Z(z)$, the pdf of $Z$.

(f) Find $P(X(0) + X(2) > 3)$.

(g) Let $Y = \int_0^{T_s} N(t) \, dt$, where $T_s$ is a constant. Find the mean $E[Y]$ and variance $E[Y^2] - (E[Y])^2$ of $Y$.

(h) Find $P(Y > 2)$.

2. Suppose that we have a signal set $\{s_i(t) : i = 1, \ldots, M\}$ defined for $t \in [0, T_s]$. We perform Gram-Schmidt orthogonalization to find an orthonormal basis $\{\phi_j(t) : j = 1, \ldots, N\}$, $N \leq M$, for the signal set. Gram-Schmidt also provides the vectors $\{s_i : i = 1, \ldots, M\}$ such that

$$s_i(t) = \sum_{j=1}^{N} s_{i,j} \phi_j(t)$$

(a) Show that

$$\int_0^{T_s} s_m(t) s_n(t) \, dt = \sum_{j=1}^{N} s_{m,j} s_{n,j}$$

for any $m$ and $n$.

(b) Show that

$$\int_0^{T_s} (s_m(t) - s_n(t))^2 \, dt = \sum_{j=1}^{N} (s_{m,j} - s_{n,j})^2$$

for any $m$ and $n$. 
3. Consider a 5-dimensional AWGN vector channel $\mathbf{z} = \mathbf{s} + \mathbf{n}$ which transmits one of the equally likely signals:

$$
\hat{\mathbf{s}}_1 = (1, 0, 1, 0, 1)^T \\
\hat{\mathbf{s}}_2 = (0, 1, 0, 1, 0)^T \\
\hat{\mathbf{s}}_3 = (0, 1, 0, -1, 0)^T \\
\hat{\mathbf{s}}_4 = (-1, -1, 0, 0, 0)^T \\
\hat{\mathbf{s}}_5 = (-1, -1, 0, -1, 0)^T \\
$$

with the orthonormal basis functions $\{\hat{\phi}_j(t) : j = 1, \ldots, 5\}$.

(a) Pick a (simple) orthonormal basis $\{\hat{\phi}_j(t) : j = 1, \ldots, 5\}$ and draw $s_1(t), s_2(t), s_3(t), s_4(t), s_5(t)$.

(b) Does this need to be a five-dimensional signal set? If your answer is no (which it will be), justify your answer by performing Gram-Schmidt on your solution to (a) to find a new orthonormal basis $\{\hat{\phi}_j(t)\}$ in fewer dimensions (Do Gram-Schmidt in the order $s_1(t), s_2(t), s_3(t), s_4(t), s_5(t)$!). Give the new signal vectors $\{\hat{\mathbf{s}}_i : i = 1, \ldots, 5\}$.

4. Consider the waveform channel:

$$
\begin{array}{c}
\begin{array}{c}
s(t) \\
\uparrow \\
n(t)
\end{array} \\
\hline \\
\begin{array}{c}
+ \\
r(t)
\end{array}
\end{array}
$$

where $n(t)$ is additive white Gaussian noise with power spectral density $\frac{N_0}{2}$. $r(t)$ is the received waveform, and $s(t) = s_i(t)$ when message $m_i$ is to be sent during time $t \in (0, T_s)$. Suppose there are $M = 4$ possible equally likely messages and the corresponding signals are:

$$
\begin{align*}
    s_1(t) &= \sqrt{\frac{2}{T_s}} \sin(2\pi f_c t) \\
    s_2(t) &= \sqrt{\frac{4}{T_s}} \cos(2\pi f_c t + \frac{\pi}{4}) \\
    s_3(t) &= -\sqrt{\frac{2}{T_s}} \sin(2\pi f_c t) \\
    s_4(t) &= \sqrt{\frac{4}{T_s}} \cos(2\pi f_c t - \frac{\pi}{4})
\end{align*}
$$

where $2\pi f_c T_s$ is an integer multiple of $2\pi$. The following identities may be useful for this problem:

$$
\begin{align*}
    \cos(A + B) &= \cos(A) \cos(B) - \sin(A) \sin(B) \\
    \cos(A - B) &= \cos(A) \cos(B) + \sin(A) \sin(B)
\end{align*}
$$
For this entire problem use the following basis functions:

\[
\phi_1(t) = \sqrt{\frac{2}{T_s}} \sin(2\pi f_c t) \\
\phi_2(t) = \sqrt{\frac{2}{T_s}} \cos(2\pi f_c t)
\]

(a) Find the signal space representation of the signals in this basis.

(b) Draw the complete (optimal) maximum a posteriori (MAP) receiver, with input \( r(t) \) and output \( \hat{m} \).

(c) Find the symbol error probability in terms of \( \frac{E_s}{N_0} \), where \( E_s \) is the average symbol energy. (Hint: If this seems unusually hard, check your answer to (a) carefully.)

(d) Find the optimal translation of the signal set to minimize the error probability of (c). How much more energy efficient (in dB) is this system than the system in part (c)?

5. [645 only] Two users each want to transmit one information bit across a pair of channels. User 1 uses both channels, while User 2 employs only the second channel so that:

\[
\begin{align*}
    r_1 &= s_1 + n_1 \\
    r_2 &= s_1 + s_2 + n_2
\end{align*}
\]

where the channel noises \( n_1 \) and \( n_2 \) are independent Gaussian random variables with mean 0 and variance \( \frac{N_0}{2} \).

The user signals are given by:

\[
\begin{align*}
    s_1 &= \begin{cases} 
        \sqrt{E_s}, & b_1 = 0 \\
        -\sqrt{E_s}, & b_1 = 1
    \end{cases} \quad \text{and} \quad
    s_2 &= \begin{cases} 
        \sqrt{E_s}, & b_2 = 0 \\
        -\sqrt{E_s}, & b_2 = 1
    \end{cases}
\end{align*}
\]

The information bits are independent of each other, and the channel noises and information bits are mutually independent. Let the receiver bit estimates for User 1 and User 2 be given by \( \hat{b}_1 \) and \( \hat{b}_2 \), respectively.

(a) Find the receiver (with inputs \( r_1, r_2 \) and outputs \( \hat{b}_1, \hat{b}_2 \)) that minimizes the probability that \( (b_1, b_2) \neq (\hat{b}_1, \hat{b}_2) \). (In other words, it maximizes the probability that both bits are correctly detected). Sketch the decision regions in terms of \( r_1 \) and \( r_2 \).

(b) Let the probability of error \( P(E) \) of the receiver in (a) be the probability that \( (b_1, b_2) \neq (\hat{b}_1, \hat{b}_2) \). Find an approximation \( \hat{P}(E) \) to this error probability such that:

\[
\lim_{\frac{E_s}{N_0} \to \infty} \frac{\hat{P}(E)}{P(E)} = 1
\]

(c) Find the equation(s) that can be solved to determine the receiver (with inputs \( r_1, r_2 \) and output \( \hat{b}_1 \)) that minimizes the probability that \( b_1 \neq \hat{b}_1 \). There is no need to simplify the equation(s), but you must give each quantity in the equation(s) explicitly.

(d) Repeat part (c), but assume that \( b_2 \) is also known at the receiver. In this case, solve the equations for the optimal receiver, and find the probability that \( b_1 \neq \hat{b}_1 \) for this receiver.
(e) Suppose that a receiver for User 1 treats the quantity \( \tilde{n} = s_2 + n_2 \) as a Gaussian random variable; in other words, the receiver for User 1 assumes that \( \tilde{n} \) is Gaussian with the same mean and variance as \( s_2 + n_2 \). Find the receiver (with inputs \( r_1, r_2 \) and output \( \hat{b}_1 \)) that minimizes the probability that \( b_1 \neq \hat{b}_1 \) under this assumption. Sketch the decision regions in terms of \( r_1 \) and \( r_2 \).