ECE 564/645 - Digital Communications, Spring 2014
Homework #2
Due: February 26, 2014 (in class)

1. Let $X(t)$ be a wide-sense stationary Gaussian random process with mean zero and autocorrelation $R_X(\tau) = e^{-\frac{|\tau|}{2}}$. Let $N(t)$ be a white Gaussian noise process with power spectral density $\frac{N_0}{2}$.

   (a) Find P_x, the power in $X(t)$.

   (b) Find $P(X(3) > 1)$.

   (c) Find the power spectral density $S_X(f)$ of $X(t)$.

   (d) Find a filter (give $h(t)$ or $H(f)$) that has input $N(t)$ and output with power spectral density $S_X(f)$.

   (e) Let $Z = X(0) + X(1) + X(2)$. Find $f_Z(z)$, the pdf of Z.

   (f) Find $P(X(0) + X(2) > 3)$.

   (g) Let $Y = \int_0^{T_s} N(t)dt$, where T_s is a constant. Find the mean $E[Y]$ and variance $E[Y^2] - (E[Y])^2$ of Y.

   (h) Find $P(Y > 2)$.

2. Suppose that we have a signal set $\{s_i(t) : i = 1, \ldots, M\}$ defined for $t \in [0, T_s]$. We perform Gram-Schmidt orthogonalization to find an orthonormal basis $\{\phi_j(t) : j = 1, \ldots, N\}$, $N \leq M$, for the signal set. Gram-Schmidt also provides the vectors $\{\underline{s}_i : i = 1, \ldots, M\}$ such that

$$s_i(t) = \sum_{j=1}^{N} s_{i,j}\phi_j(t)$$

   (a) Show that

$$\int_0^{T_s} s_m(t)s_n(t)dt = \sum_{j=1}^{N} s_{m,j}s_{n,j}$$

   for any m and n.

   (b) Show that

$$\int_0^{T_s} (s_m(t) - s_n(t))^2 dt = \sum_{j=1}^{N} (s_{m,j} - s_{n,j})^2$$

   for any m and n.

3. Consider a 5-dimensional AWGN vector channel $\underline{r} = \underline{s} + \underline{n}$ which transmits one of the equally likely signals:

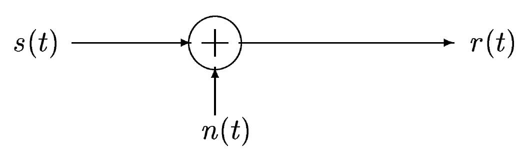
$$\begin{aligned}\tilde{\underline{s}}_1 &= (1, 0, 1, 0, 1)^T \\ \tilde{\underline{s}}_2 &= (0, 1, 0, 1, 0)^T \\ \tilde{\underline{s}}_3 &= (0, 1, 0, -1, 0)^T \\ \tilde{\underline{s}}_4 &= (-1, -1, 0, 0, 0)^T \\ \tilde{\underline{s}}_5 &= (-1, -1, 0, -1, 0)^T\end{aligned}$$

with the orthonormal basis functions $\{\tilde{\phi}_j(t) : j = 1, \ldots, 5\}$.

(a) Pick a (simple) orthonormal basis $\{\tilde{\phi}_j(t) : j = 1, \ldots, 5\}$ and draw $s_1(t)$, $s_2(t)$, $s_3(t)$, $s_4(t)$, $s_5(t)$.

(b) Does this need to be a five-dimensional signal set ? If your answer is no (which it will be), justify your answer by performing Gram-Schmidt on your solution to (a) to find a new orthonormal basis $\{\phi_j(t)\}$ in fewer dimensions (Do Gram-Schmidt in the order $s_1(t)$, $s_2(t)$, $s_3(t)$, $s_4(t)$, $s_5(t)$!). Give the new signal vectors $\{\underline{s}_i : i = 1, \ldots, 5\}$.

4. Consider the waveform channel:



where $n(t)$ is additive white Gaussian noise with power spectral density $\frac{N_0}{2}$, $r(t)$ is the received waveform, and $s(t) = s_i(t)$ when message m_i is to be sent during time $t \in (0, T_s)$. Suppose there are $M = 4$ possible **equally likely** messages and the corresponding signals are:

$$\begin{aligned}s_1(t) &= \sqrt{\frac{2}{T_s}} \sin(2\pi f_c t) \\ s_2(t) &= \sqrt{\frac{4}{T_s}} \cos(2\pi f_c t + \frac{\pi}{4}) \\ s_3(t) &= -\sqrt{\frac{2}{T_s}} \sin(2\pi f_c t) \\ s_4(t) &= \sqrt{\frac{4}{T_s}} \cos(2\pi f_c t - \frac{\pi}{4})\end{aligned}$$

where $2\pi f_c T_s$ is an integer multiple of 2π. The following identities may be useful for this problem:

$$\begin{aligned}\cos(A + B) &= \cos(A)\cos(B) - \sin(A)\sin(B) \\ \cos(A - B) &= \cos(A)\cos(B) + \sin(A)\sin(B)\end{aligned}$$

For this entire problem use the following basis functions:

$$\phi_1(t) = \sqrt{\frac{2}{T_s}} \sin(2\pi f_c t)$$
$$\phi_2(t) = \sqrt{\frac{2}{T_s}} \cos(2\pi f_c t)$$

(a) Find the signal space representation of the signals in this basis.

(b) Draw the complete (optimal) maximum a posteriori (MAP) receiver, with input $r(t)$ and output $\hat{m}$.

(c) Find the symbol error probability in terms of $\frac{E_s}{N_0}$, where E_s is the average symbol energy. (*Hint: If this seems unusually hard, check your answer to (a) carefully.*)

(d) Find the optimal translation of the signal set to minimize the error probability of (c). **How much more energy efficient (in dB) is this system than the system in part (c)?**

5. [645 only] Two users each want to transmit one information bit across a pair of channels. User 1 uses both channels, while User 2 employs only the second channel so that:

$$r_1 = s_1 + n_1$$
$$r_2 = s_1 + s_2 + n_2$$

where the channel noises n_1 and n_2 are independent Gaussian random variables with mean 0 and variance $\frac{N_0}{2}$.

The user signals are given by:

$$s_1 = \begin{cases} \sqrt{E_s}, & b_1 = 0 \\ -\sqrt{E_s}, & b_1 = 1 \end{cases} \quad \text{and} \quad s_2 = \begin{cases} \sqrt{E_s}, & b_2 = 0 \\ -\sqrt{E_s}, & b_2 = 1 \end{cases}.$$

The information bits are independent of each other, and the channel noises and information bits are mutually independent. Let the receiver bit estimates for User 1 and User 2 be given by $\hat{b}_1$ and $\hat{b}_2$, respectively.

(a) Find the receiver (with inputs r_1, r_2 and outputs $\hat{b}_1$, $\hat{b}_2$) that minimizes the probability that $(b_1, b_2) \neq (\hat{b}_1, \hat{b}_2)$. (In other words, it maximizes the probability that **both** bits are correctly detected). Sketch the decision regions in terms of r_1 and r_2.

(b) Let the probability of error $P(E)$ of the receiver in (a) be the probability that $(b_1, b_2) \neq (\hat{b}_1, \hat{b}_2)$. Find an approximation $\tilde{P}(E)$ to this error probability such that:

$$\lim_{\frac{E_s}{N_0} \to \infty} \frac{\tilde{P}(E)}{P(E)} = 1$$

(c) Find the equation(s) that can be solved to determine the receiver (with inputs r_1, r_2 and output $\hat{b}_1$) that minimizes the probability that $b_1 \neq \hat{b}_1$. There is no need to simplify the equation(s), **but you must give each quantity in the equation(s) explicitly**.

(d) Repeat part (c), but assume that b_2 is also known at the receiver. In this case, solve the equations for the optimal receiver, and find the probability that $b_1 \neq \hat{b}_1$ for this receiver.

(e) Suppose that a receiver for User 1 treats the quantity $\tilde{n} = s_2 + n_2$ as a Gaussian random variable; in other words, the receiver for User 1 assumes that $\tilde{n}$ is Gaussian with the same mean and variance as $s_2 + n_2$. Find the receiver (with inputs r_1, r_2 and output $\hat{b}_1$) that minimizes the probability that $b_1 \neq \hat{b}_1$ under this assumption. Sketch the decision regions in terms of r_1 and r_2.