ECE 603 - Probability and Random Processes, Fall 2016

Homework #2

Due: 9/30/16, in class

1. (a) Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two distinct algebras for a sample space S. Is $\mathcal{A}_1 \bigcap \mathcal{A}_2$ an algebra for the sample space S? Justify your answer with either a proof or a counterexample.

   (b) Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two distinct algebras for a sample space S. Is $\mathcal{A}_1 \bigcup \mathcal{A}_2$ an algebra for the sample space S? Justify your answer with either a proof or a counterexample.

2. I flip a fair coin twice (assume the flips are independent) and record the outcome of these flips in order. For example, for "head followed by head", the outcome is "HH". Your job is to define a probability model that will used by one of your co-workers to analyze the experiment. You are not aware of what questions he/she might want to ask, so you want to generate as complete a model as possible (e.g. you do not want to use a trivial $\mathcal{A}$). Provide a probability space $(S, \mathcal{A}, P)$ for this experiment. Since the size of the sets involved here is not that large, be explicit about how each of these three things are defined. In particular, write out all of the sets in $\mathcal{A}$ and give the probability of each.

3. Consider the probability space $(S, \mathcal{A}, P)$, with $S = [0, 1]$ and $\mathcal{A} = \mathcal{B}$ (restricted to $[0, 1]$,of course), and $P(\cdot)$ defined as follows:

$$P((a,b)) = \begin{cases} c \cdot ((b-a) - \frac{1}{2}(b^2 - a^2)), & 0 \le a < b \le \frac{1}{2} \\ c \cdot ((b-a) - \frac{1}{2}(b^2 - a^2)) & \frac{1}{2} \le a < b \le 1 \\ \frac{1}{2} + c \cdot ((b-a) - \frac{1}{2}(b^2 - a^2)), & a < \frac{1}{2} < b \le 1 \end{cases}$$

   (a) Find c.

   (b) Find $P(X = 1/4)$.

   (c) Find the probability that the outcome is an irrational number.

4. Suppose you take an exam with three questions, which have the following point values:

   Question 1: 20 points
   Question 2: 10 points
   Question 3: 10 points

   On this fictitious exam, there is no partial credit for a given question; hence, you either get the question correct (full points), or you get it wrong (no points).

   The probability you get Question 1 correct is 0.90. Unfortunately, Question 2 is dependent on Question 1: if you get Question 1 correct, there is a probability of 0.9 that you will also get Question 2 correct. However, if you get Question 1 wrong, there is a probability of 0.3 that you will get Question 2 correct. Question 3 is independent of Questions 1 and 2, and you get Question 3 correct with probability 0.5.

Let Q_i be the event you get Question i correct.

(a) Find the probability that you get Question 2 correct; in other words, find $P(Q_2)$.

(b) Given that you got Question 2 correct, find the probability that you got Question 1 correct.

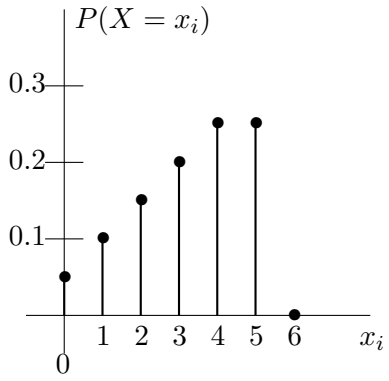
(c) Given that you got Question 3 correct, find the probability that you got Question 1 correct.

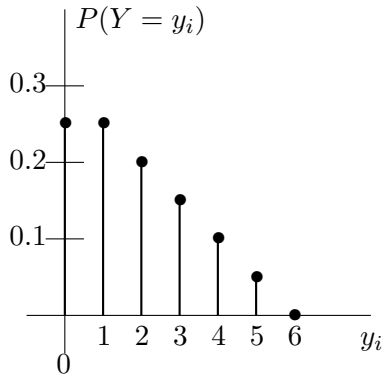
(d) Suppose that you pass the exam if you score greater than or equal to 30 points. Let A be the event that you pass the exam.

- Write A in terms of Q_1, Q_2, and Q_3.
- Find $P(A)$.

5. *This problem looks like a random variable problem (for those of you who know what that is), but the whole point is that you are able to solve it using what you know already if you can read a graph.*

   Let X be the number of bad chips in a shipment shipped from Xcompany, and let Y be the number of bad chips in a shipment shipped from Ycompany. The reliability sheets from the two companies give the probability of a given number of bad chips in a given order. They display their information graphically, as follows:





   For example, the probability of getting no bad chips in a shipment from Xcompany is 0.05, the probability of getting one bad chip in a shipment from Xcompany is 0.1, the probability of getting two bad chips in a shipment from Xcompany is 0.15, etc.

   (a) Suppose you receive a shipment of chips from Xcompany. Find the probability that the number of bad chips is less than or equal to 3.

   (b) Suppose you receive a shipment of chips from Xcompany. Somebody tells you that there are at

least two bad chips in the shipment. Given this information, find the probability that the number of bad chips is less than or equal to 3.

(c) Suppose a shipment of chips arrives, but you are not sure from which company it came; however, you know the probability it came from Xcompany is 0.25 and the probability it came from Ycompany is 0.75. Find the probability that the number of bad chips in the shipment is less than or equal to 3.

(d) Before the shipment arrives, you know the probability it is coming from Xcompany is 0.25 and the probability it is coming from Ycompany is 0.75. Suppose you receive the shipment and figure out that there are 3 bad chips in the shipment. Given this additional information, what is the probability the shipment came from Xcompany?

(e) If you could choose to buy chips from only one company (X or Y), from which company would you buy chips? **Be sure to justify your answer.**

6. [The famous (and very difficult) "Oscar and his dog" problem. Good luck!] Oscar has lost his dog in forest A (with probability 0.4) or in forest B (with probability 0.6). If the dog is alive and not found by the N^{th} day of the search, it will die that evening with probability $\frac{N}{N+2}$.

   If the dog is in A (either dead or alive) and Oscar spends a day searching for the dog in A, the probability that he will find the dog that day is 0.25. Similarly, if the dog is in B and Oscar spends a day looking for it there, he will find the dog that day with probability 0.15.

   Rules: The dog cannot go from one forest to the other. Oscar can search only in the daytime, and he can travel from one forest to the other only at night. All parts of this problem are to be worked separately.

   (a) In which forest should Oscar look to maximize the probability he finds his dog on the first day of the search?

   (b) Given that Oscar looked in A on the first day but did not find his dog, what is the probability that the dog is in A?

   (c) If Oscar flips a fair coin to determine where to look on the first day and finds his dog on the first day, what is the probability that he looked in A?

   (d) Oscar has decided to look in A for the first two days. What is the probability that he will find a live dog for the first time on the second day?

   (e) Oscar has decided to look in A for the first two days. Given the fact that he was unsuccessful on the first day, determine the probability that he does not find a dead dog on the second day.

   (f) Oscar finally found his dog on the fourth day of the search. He looked in A for the first 3 days and in B on the fourth day. What is the probability that he found his dog alive?