1. Your buddy who works in the modeling department provides you with the following probability space:

\[ S = (0, 1), \mathcal{A} = \mathcal{B} \text{ (where the Borel field is restricted to (0, 1), of course) and, for any interval,} \]

\[ P((a, b)) = \begin{cases} 
\frac{b^2 - a^2}{2}, & 0 \leq a < b \leq \frac{1}{2} \\
\frac{b^2 - a^2}{2}, & \frac{1}{2} \leq a < b \leq 1 \\
\frac{1}{2} + \frac{b^2 - a^2}{2}, & a < \frac{1}{2} < b \leq 1 
\end{cases} \]

(a) Find the probability of the outcome \( \frac{1}{4} \).

(b) Find the probability of the set of irrational numbers in \((0, 1)\).

2. In a box there are 20 parts, 3 of which are defective. You draw parts from the box one after another without replacement and test them to find the defective ones. Find the probability of the following:

(a) You have drawn and tested 10 parts and you have not observed any defective parts.

(b) Within the 10 parts drawn/tested, you have observed 2 defective parts.

(c) On the 10th draw/test, you observe the third defective part.

3. Suppose that three trucks leave a warehouse with 20 parts in the first truck, 40 parts in the second truck, and 40 parts in the third truck. Suppose that there are 5, 10, and 2 defective parts out of those in the respective trucks. Suppose I receive my shipment of five parts randomly drawn from those on one of the trucks, where the three trucks are equally likely to have made my delivery.

(a) What is the probability that there are exactly three defective parts out of the five parts in my shipment?

(b) Given that there are exactly three defective parts in my shipment, what is the probability that it came off of the second truck.
4. Let $X$ be the number of bad chips in a shipment shipped from Xcompany, and let $Y$ be the number of bad chips in a shipment shipped from Ycompany. The reliability sheets from the two companies give the probability of a given number of bad chips in a given order. They display their information graphically, as follows:

For example, the probability of getting no bad chips in a shipment from Xcompany is 0.05, the probability of getting one bad chip in a shipment from Xcompany is 0.1, the probability of getting two bad chips in a shipment from Xcompany is 0.15, etc.

(a) Suppose you receive a shipment of chips from Xcompany. Find the probability that the number of bad chips is less than or equal to 3.

(b) Suppose you receive a shipment of chips from Xcompany. Somebody tells you that there are at least two bad chips in the shipment. Given this information, find the probability that the number of bad chips is less than or equal to 3.

5. Consider the probability space $((0,2), B, P)$, where $B$ is restricted to $(0,2)$, of course, and $P(\cdot)$ is defined by:

$$P((a,b)) = \begin{cases} c (b^2 - a^2), & 0 \leq a < b \leq \frac{2}{3} \\ c (b^2 - a^2), & \frac{2}{3} \leq a < b \leq 2 \\ \frac{3}{4} + c (b^2 - a^2), & a < \frac{2}{3} < b \leq 2 \end{cases}$$

where $c$ is a constant. Let $X$ be the outcome of the experiment.

(a) What is $P(\frac{1}{2} < X < 1)$? Your answer should be a number.

(b) What is $P(\frac{2}{3} \leq X < 2)$? Be sure to derive everything from first principles!

(c) Since $X \in \mathcal{R}$, it can be treated as a random variable. Find and roughly sketch the cumulative distribution function (CDF) $F_X(x)$ and probability density function $f_X(x)$ of $X$.

(d) Let the random variable $Y$ be defined as $Y = X^2 - 2$. Find the probability space $(S, \mathcal{A}, P_Y)$ for $Y$. Hint: This looks like a “function of random variable” problem, which we have not studied yet, but you can still easily solve it from first principles.
6. A salesman visits one of three cities: X-ville, Y-ville, and Z-ville. When he visits a given city, the corresponding probability density function (pdf) of the money that he obtains is given by:

\[
\begin{align*}
  f_X(x) &= \frac{1}{4} \delta(x) + \frac{1}{2} \delta(x - 5) + \frac{1}{4} \delta(x - 10) \\
  f_Y(y) &= \frac{1}{2} \delta(y) + \frac{1}{2} \delta(y - 10) \\
  f_Z(z) &= \begin{cases} \\
    1/10, & 0 \leq z \leq 10 \\
    0, & \text{otherwise} \\
  \end{cases}
\end{align*}
\]

(a) Suppose he chooses a city at random to visit. Find the probability that he makes greater than or equal to $5.

(b) Suppose he chooses a city at random to visit. Given that he makes greater than or equal to $5, find the probability that he visited city X.

(c) Any of the cities can claim that they are the “best” city for the salesman to obtain money - if they use the correct argument. Give the argument that each can make. In other words, for each city, give a measure by which it is the “best”.

(d) Suppose he does 20 visits to city Y, and the money obtained for each visit is independent of any other visit. Write an expression for the probability that he makes more than $150. (Your expression should only contain simple terms that are easily evaluated.)