ECE 603 - Probability and Random Processes, Fall 2016

Homework #1

Due: 09/23/16, in class

1. [Pishro-Nik, Chapter 1, Problem 10] Consider the set of natural numbers $\mathcal{N} = \{0, 1, 2, \ldots, \}$. Let $\mathcal{Y}$ be the set of all subsets of $\mathcal{N}$. Is $\mathcal{Y}$ countable or uncountable? (Note: If you look at the problem description in the Pishro-Nik book, there are hints that will help you.)

2. Determine whether the following sets are finite, countably infinite, or uncountable (*Hint: Think of a function as a table showing the output for every possible input. For example, a function $f : \{0, 1\} \longrightarrow \mathcal{Z}_+$ can be specified by a two-element table containing the values for $f(0) \in \mathcal{Z}_+$ and $f(1) \in \mathcal{Z}_+$. A sample function/table might be $f(0) = 32$, $f(1) = 221$. Then, think how many distinct tables there could be.*):

   (a) The set A of all functions $f : \{0, 1\} \longrightarrow \mathcal{Z}_+$

   (b) The set B_n of all functions $f : \{1, 2, \ldots, n\} \longrightarrow \mathcal{Z}_+$

   (c) The set $C = \bigcup_{n \in \mathcal{Z}_+} B_n$.

   (d) The set D of all functions $f : \mathcal{Z}_+ \longrightarrow \{0, 1\}$.

   (e) The set E of all functions $f : \mathcal{Z}_+ \longrightarrow \mathcal{Z}_+$.

   (f) The set F of all functions $f : \mathcal{Z}_+ \longrightarrow \{0, 1\}$ such that $f(n) = 0$ for $n \geq N$, where N is some integer.

   (g) The set G of all functions $f : \mathcal{Z}_+ \longrightarrow \mathcal{Z}_+$ such that $f(n) = 1$ for $n \geq N$, where N is some integer.

   (h) The set H of all functions $f : \mathcal{Z}_+ \longrightarrow \mathcal{R}$ such that $f(n) = 1$ for $n \geq N$, where N is some integer.

   (i) The set I of all two-element subsets of $\mathcal{Z}_+$.

   (j) The set J of all finite subsets of $\mathcal{Z}_+$.

3. A number is chosen at random from the interval [0,1]. As is the standard case, the probabilities are defined on the Borel σ-algebra (restricted to [0,1]). **Starting from first principles** (i.e. definition of the Borel σ-algebra, axioms of probability, etc.), answer the following three parts:

   (a) Let A be a subset of $[0, 1]$ that is **not** in the Borel σ-algebra. Show that A must contain an uncountable number of elements.

   (b) Let D be an arbitrary uncountable subset of $[0, 1]$. Is $\overline{D}$, the complement of D, necessarily countable?

   (c) Let C be the set of irrational numbers in $[0, 1]$; that is, $x \in C$ if and only if $0 \leq x \leq 1$ and $x \neq \frac{m}{n}$ for all $m \in \{0, 1, 2, 3, \ldots\}$, $n \in \{0, 1, 2, 3, \ldots\}$. Find the probability of C.

4. Suppose that I conduct the following experiment. I flip a coin continually, and I write down the result of the flips to get an outcome ω. For example, a possible ω might look like:

$$\omega = (\text{Tail, Tail, Tail, Head, Head}, \ldots)$$

(a) Consider the sample space S of all possible outcomes ω. Is S countable or uncountable?

(b) Construct a rich (i.e. non-trivial) probability space $(S, \mathcal{A}, P)$ for this experiment. You should end up with an $\mathcal{A}$ with an uncountable number of elements that should allow one to answer any question of interest.

(c) Show that the following events are in your probability space, and, **using your probability space, find their probability**:

- The event that I get the outcome that alternates Tails and Heads forever: $\omega =$ (Tail, Head, Tail, Head, etc.).
- The event that the first three flips are Tail.
- The event that flips four through six are Tail.

5. I have analyzed two *independent* experiments, Experiment 1 and Experiment 2, to arrive at two separate probability spaces: $(\Omega_1, \mathcal{A}_1, P_1)$ and $(\Omega_2, \mathcal{A}_2, P_2)$, where:

   $\Omega_1 = \{1, 2\}$, $\mathcal{A}_1 = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$, and $P_1(\cdot)$ is defined by $P_1(\phi) = 0$, $P_1(\{1\}) = 0.4$, $P_1(\{2\}) = 0.6$, $P_1(\Omega) = 1$.

   $\Omega_2 = \{2, 4\}$, $\mathcal{A}_2 = \{\phi, \{2\}, \{4\}, \{2, 4\}\}$, and $P_2(\cdot)$ is defined by $P_2(\phi) = 0$, $P_2(\{2\}) = 0.2$, $P_2(\{4\}) = 0.8$, $P_2(\Omega) = 1$.

   (a) Are these valid probability spaces? *Be sure to tell me all of the conditions that you checked to arrive at your answer.*

   (b) My boss asks me to define a combined experiment as follows: Perform Experiment 1 and remember the result; then, perform Experiment 2 and remember the result. Now, write in your notebook the *outcome* of the combined experiment as an ordered pair with the first entry equal to the result of Experiment 1 and and the second entry equal to the result of Experiment 2. For example, an outcome might be "(1,4)". Find $(\Omega, \mathcal{A}, P)$ for the combined experiment. Use a $\mathcal{A}$ that captures as many events as possible, and be sure to write out explicitly at least half of the events in $\mathcal{A}$.

   (c) Alas, the boss is fickle and changes his mind. Now he asks: perform Experiment 1 and remember the result; then, perform Experiment 2 and remember the result. Now, write in your notebook the *outcome* of the combined experiment as a *random variable* X equal to the result of Experiment 1 times the result of Experiment 2. For example, an outcome might be "4" (which is 2×2). Find $(\Omega, \mathcal{A}, P)$ for the random variable X. *Hint: Feel free to define P using the integral of a function if this makes it easier to represent.*

   Now, your buddy in the modeling department comes to you with yet another experiment description: $(\Omega_3, \mathcal{A}_3, P_3)$, where
   $\Omega_3 = \{1, 2, 3\}$, $\mathcal{A}_3 = \{\phi, \{1\}, \{2, 3\}, \{1, 2, 3\}\}$, and $P_3(\cdot)$ is defined by $P(\phi) = 0, P(\{1\}) = 0.1, P(\{2, 3\}) = 0.9, P(\{1, 2, 3\}) = 1$.

   (d) Is $(\Omega_3, \mathcal{A}_3, P_3)$ a valid probability space? *Be sure to tell me all of the conditions that you checked to arrive at your answer.*

   (e) Your boss asks you to use the description of $(\Omega_3, \mathcal{A}_3, P_3)$ to find the probability that a "3" is observed. How do you respond?