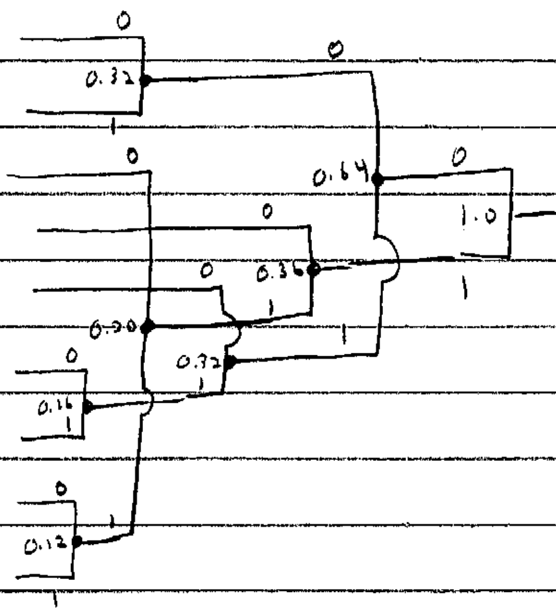
Final Exam Solutions

ECE 564/645 Spring 2019

1)(a)

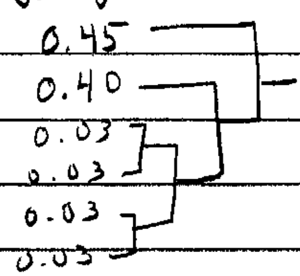
		code	p(x) l(x)
AA	0.16	000	0.48
AB	0.16	001	0.48
AC	0.08	110	0.24
BA	0.16	10	0.32
BB	0.16	010	0.48
BC	0.08	0110	0.32
CA	0.08	0111	0.32
CB	0.08	1110	0.32
CC	0.04	1111	0.16
			3.12



$$R = 3.12/2 = 1.56 \text{ bits/symbol}$$

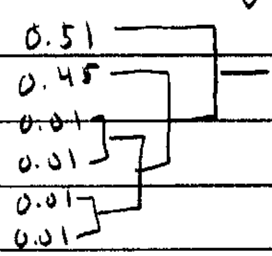
not on exam (removed)

Certainly, you could have a 1-bit sequence



$\Rightarrow$ lower bound = 1

Could it be larger?



could be 2

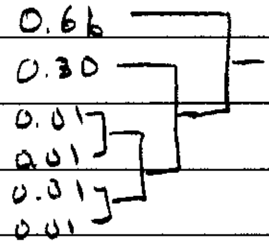
But, being the second largest -and- all others (except largest) sum to $\leq 0.45 \Rightarrow \ell = 2$ is max $1 \leq \ell(x) \leq 2$

(b) Can it $\ell = 1$?

No.

When there are three blocks left, 0.30 <u>cannot</u> be the largest.

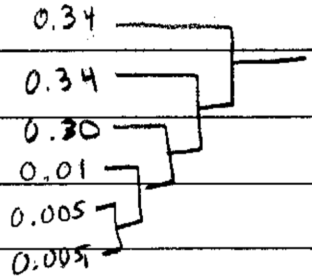
Can I have $\ell = 2$? yes



lower bound = 2

Upper bound?

At most the third largest block



$\ell \leq 3$

$2 \leq \ell(x) \leq 3$

2) (a) $\int_0^1 \cos(2\pi t)\,dt = \frac{1}{2\pi}\sin(2\pi t)\Big|_0^1 = 0 - 0 = 0$

$\int_0^1 \sin(2\pi t)\,dt = \frac{-1}{2\pi}\cos(2\pi t)\Big|_0^1 = -1+1 = 0$

$\int_0^1 \sin(2\pi t)\cos(2\pi t)\,dt = \int_0^1 \frac{1}{2}\sin(4\pi t)\,dt = \frac{-1}{8\pi}\cos(8\pi t) = -1+1 = 0$

$\int_0^1 \cos^2(2\pi t)\,dt = \int_0^1 \left(1/2 + 1/2\cos(2\pi t)\right)dt$

$= 1/2 + 1/4\pi \sin(2\pi t)\Big|_0^1 = 1/2$

(b) From part (a), 1, $\cos(2\pi t)$, and $\sin(2\pi t)$ form an orthogonal set. Thus, $N \geq 3$.

Try:

$\phi_1(t) = 1, \quad 0 \le t \le 1$

$\phi_2(t) = \sqrt{2}\cos(2\pi t), \quad 0 \le t \le 1$

$\phi_3(t) = \sqrt{2}\sin(2\pi t), \quad 0 \le t \le 1$

All four signals can be written in terms of $\phi_1(t), \phi_2(t), \phi_3(t)$.

$\underline{s}_1 = (2, 0, 0)^T$

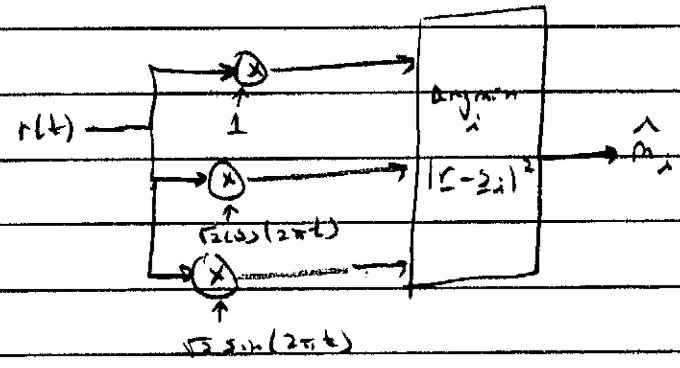
$\underline{s}_2 = (1, 2, 0)^T$

$\underline{s}_3 = (1, 0, 2)^T$

$\underline{s}_4 = (-1, 0, 2)^T$

Also, could ~~do G-S~~ easily, using (a)

(c)



(d)

$$P(E) \le \frac{1}{m} \sum_{i=1}^{m} \sum_{j \ne i} Q\left(\frac{d_{ij}}{\sqrt{2N_0}}\right)$$

	1	2	3	4
1	–	$\sqrt{5}$	$\sqrt{5}$	$\sqrt{13}$
2	$\sqrt{5}$	–	$\sqrt{8}$	$\sqrt{12}$
3	$\sqrt{5}$	$\sqrt{8}$	–	2
4	$\sqrt{13}$	$\sqrt{12}$	2	–

$$\le \frac{1}{4}\left(4Q\left(\sqrt{\frac{5}{2N_0}}\right) + 2Q\left(\sqrt{\frac{4}{N_0}}\right) + 2Q\left(\sqrt{\frac{13}{2N_0}}\right) + 2Q\left(\sqrt{\frac{6}{N_0}}\right) + \underbrace{2Q\left(\sqrt{\frac{2}{N_0}}\right)}_{\text{dominant term}}\right)$$

$$E_s = \frac{1}{4}(4 + 5 + 5 + 5)$$

$$= 19/4$$

$$\frac{1}{2} Q\left(\sqrt{\frac{2}{N_0}}\right) = 1/2\, Q\left(\sqrt{\frac{2 \cdot 4/19 \cdot 19/4}{N_0}}\right) = \frac{1}{2} Q\left(\sqrt{\frac{8E_s}{19N_0}}\right)$$

[and

$E_s = E_b \log_2 m = 2E_b$]

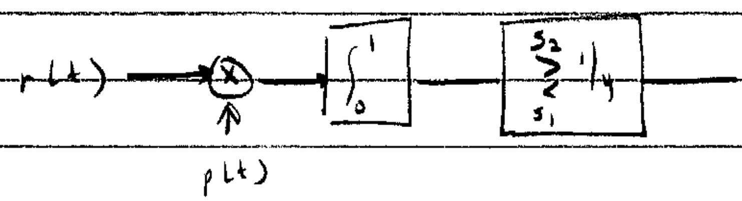
$$= \frac{1}{2} Q\left(\sqrt{\frac{16E_b}{19N_0}}\right)$$

3) (a)

$$\phi(t) = p(t)$$

$$\underline{s}_1 = -1/4$$

$$\underline{s}_2 = 3/4$$



(b) $$P_b = Q\left(d/\sqrt{2N_0}\right) = Q\left(1/\sqrt{2N_0}\right) = Q\left(\sqrt{\frac{16/5 \cdot 5/16}{2N_0}}\right) = Q\left(\sqrt{\frac{8}{5} \cdot \frac{E_s}{N_0}}\right)$$

$$E_s = 1/2\left((1/4)^2 + (3/4)^2\right) = 5/16$$

(c) $$10 \log_{10}\left(\frac{2}{8/5}\right) = 10 \log_{10} 5/4 \text{ dB worse!}$$

(d) Need three linearly independent codewords:

$$G = \overbrace{\begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}}^{n=6} \Big\} k=3 \qquad r = k/n = 3/6 = 1/2$$

$$\mathcal{C} = \{000000, 001110, 110010, 111100, 101001, 100111, 011011, 010101\}$$

$$A(x) = 1 + 4x^3 + 3x^4$$

s	leader
000	000000
001	000001
010	000010
011	000011
100	001000
101	100000
110	000100
111	010000

(e) $$p = Q\left(\sqrt{\tfrac{8}{5} E_s/N_0}\right) = Q\left(\sqrt{\tfrac{4}{5} E_b/N_0}\right)$$

$$r E_b = E_s$$

$$E_b = 2E_s$$

and $$P(E) = 1 - P(c) = 1 - (1-p)^6 - 6p(1-p)^5 - p^2(1-p)^4$$

(f) $$P(E) \le \sum_{i \neq 0} Q\left(\frac{d_{0,i}}{\sqrt{2N_0}}\right)$$

$$= 4Q\left(\sqrt{\frac{3 \cdot 1^2}{2N_0}}\right) + 3Q\left(\sqrt{\frac{4 \cdot 1^2}{2N_0}}\right)$$

$$= 4Q\left(\sqrt{\frac{3 \cdot 16/5 \cdot 5/16}{2N_0}}\right) + 3Q\left(\sqrt{\frac{4 \cdot 16/5 \cdot 5/16}{2N_0}}\right)$$

$$= 4Q\left(\sqrt{\frac{24}{5}\frac{E_s}{N_0}}\right) + 3Q\left(\sqrt{\frac{32}{5}\frac{E_s}{N_0}}\right)$$

$[E_b = 2E_s]$

$$= 4Q\left(\sqrt{\tfrac{12}{5} E_b/N_0}\right) + 3Q\left(\sqrt{\tfrac{16}{5} E_b/N_0}\right)$$

4) (a)

For a t-error correcting code, there are

$$1 + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{t}$$

patterns to be corrected. Each must have its own syndrome; thus,

codewords

$$2^{n-k} \geq 1 + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{t}$$

$$\Rightarrow 2^k \leq 2^n \Big/ \left(1 + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{t}\right)$$

(b)

$$H = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$k=4$, $n=7$, $t=1$

$$3 = \log_2(1+7) = 3 \quad \checkmark$$