**ECE 564/645 - Digital Communication Systems (Spring 2014)**

**Final Exam**

**Friday, May 2nd, 8:00-10:00am, Marston 220**

## Overview

- The exam consists of four (or five) problems for 100 (or 120) points. The points for each part of each problem are given in brackets - you should spend your **two hours** accordingly.
- The exam is closed book, but you are allowed **three page-sides** of notes. Calculators are **not** allowed. I will provide all necessary blank paper.

## Testmanship

- **Full credit will be given only to fully justified answers.**
- Giving the steps along the way to the answer will not only earn full credit but also maximize the partial credit should you stumble or get stuck. If you get stuck, attempt to neatly define your approach to the problem and why you are stuck.
- If part of a problem depends on a previous part that you are unable to solve, explain the method for doing the current part, and, if possible, give the answer in terms of the quantities of the previous part that you are unable to obtain.
- Start each problem on a new page. Not only will this facilitate grading but also make it easier for you to jump back and forth between problems.
- If you get to the end of the problem and realize that your answer must be wrong (e.g. a negative probability), be sure to write "this must be wrong because ..." so that I will know you recognized such a fact.
- Academic dishonesty **will** be dealt with **harshly** - the *minimum penalty* will be an "F" for the course.

## Some potentially useful information

$$\begin{aligned}\cos(\theta) &= \tfrac{1}{2}\left(e^{j\theta}+e^{-j\theta}\right)\\ \sin(\theta) &= \tfrac{1}{2j}\left(e^{j\theta}-e^{-j\theta}\right)\end{aligned}$$

$$\begin{aligned}\sin(a\pm b) &= \sin(a)\cos(b)\pm\cos(a)\sin(b)\\ \cos(a\pm b) &= \cos(a)\cos(b)\mp\sin(a)\sin(b)\end{aligned}$$

$$\begin{aligned}\cos(a)\cos(b) &= \tfrac{1}{2}\left[\cos(a-b)+\cos(a+b)\right]\\ \sin(a)\sin(b) &= \tfrac{1}{2}\left[\cos(a-b)-\cos(a+b)\right]\\ \sin(a)\cos(b) &= \tfrac{1}{2}\left[\sin(a-b)+\sin(a+b)\right]\end{aligned}$$

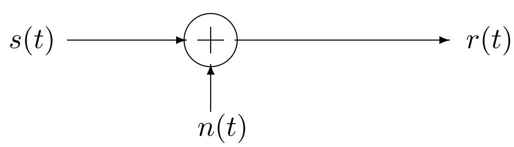
$$\begin{aligned}\cos^2(a) &= \tfrac{1}{2}\left[1+\cos(2a)\right]\\ \sin^2(a) &= \tfrac{1}{2}\left[1-\cos(2a)\right]\\ \sin(a)\cos(a) &= \tfrac{1}{2}\sin(2a)\end{aligned}$$

1. *Huffman Coding:*

   [12] (a) A source produces an independent and identically distributed (IID) sequence from $\mathcal{X} = \{A, B, C\}$, with $P(X_i = A) = 0.4, P(X_i = B) = 0.4, P(X_i = C) = 0.2$. An optimal lossless source coder takes symbols $N = 2$ at a time and compresses this source at *minimum rate*. Find such a source code **and** its rate (in output bits per input symbol).

   [8] (b) Suppose I have a source that produces a random variable $X$ from an alphabet $\mathcal{X} = \{A, B, C, D, E, F\}$, and consider coding this single random variable with an optimal lossless source coder (i.e. Huffman coding with $N = 1$). My friend knows the probabilities of each of the letters (which he does not tell me) and designs such a Huffman code. He then tells me that $P(X = A) = 0.3$, but not any of the other probabilities, and "$A$" may or may not be the most likely symbol. Give upper and lower bounds on the length of the bit sequence assigned to "$A$" by the Huffman code based on this information. **Be sure to justify your answer.** (*Hint: Analyze the Huffman coding algorithm.*)

2. Consider the waveform channel:




   where $n(t)$ is additive white Gaussian noise with power spectral density $\frac{N_0}{2}$, $r(t)$ is the received waveform, and $s(t) = s_i(t)$ when message $m_i$ is to be sent during time $t \in (0, 1)$. Suppose there are $M = 4$ possible **equally likely** messages and the corresponding signals are:

$$\begin{aligned} s_1(t) &= 2, & 0 \le t \le 1 \\ s_2(t) &= 1 + 2\sqrt{2}\cos(2\pi t), & 0 \le t \le 1 \\ s_3(t) &= 1 + 2\sqrt{2}\sin(2\pi t), & 0 \le t \le 1 \\ s_4(t) &= -1 + 2\sqrt{2}\sin(2\pi t), & 0 \le t \le 1 \end{aligned}$$

   [7] (a) Calculate the integrals:

   - $\int_0^1 \cos(2\pi t)dt$
   - $\int_0^1 \sin(2\pi t)dt$
   - $\int_0^1 \sin(2\pi t)\cos(2\pi t)dt$
   - $\int_0^1 \cos^2(2\pi t)dt$

   [13] (b) Find an orthonormal basis $\{\phi_i(t) : i = 1, 2, \ldots, N\}$ for this signal set **and** give the vector representation of each signal in this basis.

   [5] (c) Specify the MAP receiver by:

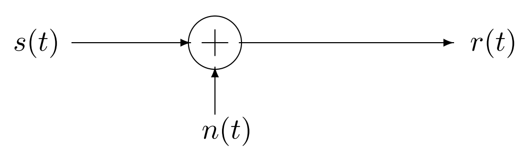
   - Drawing a (simple) block diagram showing a method of obtaining $r_j$ (the component of $r(t)$ along the basis function $\phi_j(t)$) from $r(t)$.
   - Indicating how a signal is chosen based on ($\underline{r} = (r_1, \ldots, r_N)^T$), (You can do this with a picture or a description with equations.)

   [10] (d) Find the Union Bound to the probability of error of the MAP receiver in terms of units of your signal space. Identify the term in your sum that will dominate performance at moderate-to-high signal-to-noise ratios. **Convert this dominant summand only** to be in terms of $E_b$ and $N_0$.

3. Consider the following (unusual) binary amplitude-shift keying (ASK) system for transmitting a bit $b_0$, **equally likely** to be 0 or 1, in $t \in (0,1)$. For $b_0 = 0$, we let $s(t) = -\frac{1}{4}p(t)$, and for $b_0 = 1$, we let $s(t) = \frac{3}{4}p(t)$, where

$$p(t) = \begin{cases} 1 & 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

The signal $s(t)$ is transmitted across a channel modeled as the following:




where $n(t)$ is additive white Gaussian noise with power spectral density $\frac{N_0}{2}$ and $r(t)$ is the received waveform.

[5] (a) Find the receiver for processing $r(t), t \in (0,1)$ to obtain an estimate for the transmitted bit that minimizes the probability of a bit error.

[5] (b) Find the probability of a bit error in terms of the average energy per symbol $E_s$ and $N_0$.

[5] (c) How much better or worse (in dB of $\frac{E_s}{N_0}$) is this system than a binary phase-shift keyed (BPSK) system operating on an AWGN channel?

Suppose now that channel coding is employed in the system; that is, the information bits $\{I_k\}$ are input to the encoder to produce the bits $\{b_k\}$ for modulation. The parity check matrix of the code is given by:

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

[15] (d) Find the following:

- A generator matrix $G$ for the code.
- The weight enumerator polynomial for the code: $A(x) = \sum_{i=0}^{n} A_i x^i$, where $A_i$ is the number of weight-$i$ codewords.
- The rate $r$ of the code.
- The syndromes and the corresponding coset leaders.

[7] (e) Now suppose the bits from the encoder are transmitted with the binary amplitude shift-keyed (ASK) system on an AWGN channel as described in the first part of the problem. Find the **exact** probability of a codeword error in terms of $E_s$ and $N_0$ when **hard-decision decoding** is done at the receiver; that is, the demodulator outputs its best estimate of each bit $b_k$ and the decoder uses six such bit decisions to decode the codeword. **Then,** convert your answer to be in terms of the energy *per information bit* (i.e. energy per bit at the input to the channel coder) $E_b$ and $N_0$.

[8] (f) Now suppose the bits from the encoder are transmitted with the binary amplitude shift-keyed (ASK) system on an AWGN channel as described in the first part of the problem. Find the Union Bound to the probability of a codeword error in terms of $E_s$ and $N_0$ when **optimal soft-decision decoding** is done at the receiver; that is, the decoder chooses the most likely codeword transmitted based on the received waveform over the corresponding six symbol periods. **Then,** convert your answer to be in terms of the energy *per information bit* (i.e. energy per bit at the input to the channel coder) $E_b$ and $N_0$.

4. [645 only] When decoding a linear block code, we can employ the standard array, and we know an error pattern is correctible if and only if it is declared a coset leader.

[10] (a) Consider a $t$-error correcting $(n, k)$ linear block code. (*Recall, that for a code to be $t$-error correcting, it must correct* **every** *pattern of t or fewer errors.*) Show that the following equation must be true:

$$n - k \geq \log_2 \left( 1 + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{t} \right)$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. **Then,** show how this implies an upper bound on the number of codewords in a code for a given $t$ and $n$.

[10] (b) Without using a repetition code, find an example of a linear block code (give $G$, $H$, or the codewords) with $t \geq 1$ (you get to choose $t$ as long as $t \geq 1$) such that:

$$n - k = \log_2 \left( 1 + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{t} \right)$$