ECE 645 - Digital Communication Systems (Spring 2013)

Final Exam

Monday, May 6th, 1:30-3:30pm, Marston 211

Overview

- The exam consists of four problems for 100 points. The points for each part of each problem are given in brackets **-** you should spend your **two hours** accordingly.
- The exam is closed book, but you are allowed **three page-sides** of notes. Calculators are **not** allowed. I will provide all necessary blank paper.

Testmanship

- **Full credit will be given only to fully justified answers.**
- Giving the steps along the way to the answer will not only earn full credit but also maximize the partial credit should you stumble or get stuck. If you get stuck, attempt to neatly define your approach to the problem and why you are stuck.
- If part of a problem depends on a previous part that you are unable to solve, explain the method for doing the current part, and, if possible, give the answer in terms of the quantities of the previous part that you are unable to obtain.
- Start each problem on a new page. Not only will this facilitate grading but also make it easier for you to jump back and forth between problems.
- If you get to the end of the problem and realize that your answer must be wrong (e.g. a negative probability), be sure to write "this must be wrong because ..." so that I will know you recognized such a fact.
- Academic dishonesty **will** be dealt with **harshly -** the *minimum penalty* will be an "F" for the course.

1. *Huffman Coding:*

   [10] (a) Suppose that you have an independent and identically distributed (IID) source sequence $\{Y_n\}$, which takes values from an alphabet $\mathcal{Y} = \{A, B, C, D, E, F\}$ according to some probability distribution. Suppose someone comes to you and tells you they have designed a Huffman code for this source which takes $N = 2$ symbols at a time, and the resulting Huffman code has rate $R = 3$ bits per (input) symbol. Can such a Huffman code exist? (*Be sure to justify your answer.*)

   [10] (b) An independent and identically distributed (IID) binary source sequence $\{X_n\}$ has $P(X_n = 0) = 0.8$, $P(X_n = 1) = 0.2$, $\forall n$. Suppose that I take blocks of $N = 2$ bits at a time from this source as the input to my lossless source coder. Find a Huffman code for this situation (i.e. a Huffman code that assigns to each 2-bit input sequence a variable-length output bit sequence), and find the rate of your code in average output bits per input bit. (*Note: This should be short and easy.*)

   [10] (c) Now, your boss comes to you and tells you that the source from (b) is still stationary but now it is correlated. In fact, he tells you that, for any n, the conditional probability distribution of the n^{th} symbol only depends on the previous symbol; that is,

$$p_{X_n|X_1,X_2,X_3,\ldots,X_{n-1}}(x_n|x_1,x_2,x_3,\ldots,x_{n-1}) = p_{X_n|X_{n-1}}(x_n|x_{n-1}) = \begin{cases} 0.9, & x_n = 0, x_{n-1} = 0 \\ 0.1, & x_n = 1, x_{n-1} = 0 \\ 0.4, & x_n = 0, x_{n-1} = 1 \\ 0.6, & x_n = 1, x_{n-1} = 1 \end{cases}$$

   Using stationarity and this conditional probability, you can show that the marginal probabilities are still $P(X_n = 0) = 0.8$, $P(X_n = 1) = 0.2$. Design a Huffman code that works on blocks of $N = 2$ input bits and find the rate of your code in average output bits per input bit.

2. [10] Suppose that we have a signal set $\{s_i(t) : i = 1, \ldots, M\}$ defined for $t \in [0, T_s]$. We perform Gram-Schmidt orthogonalization to find an orthonormal basis $\{\phi_j(t) : j = 1, \ldots, N\}$, $N \leq M$, for the signal set. Gram-Schmidt also provides the vectors $\{\underline{s}_i : i = 1, \ldots, M\}$ such that

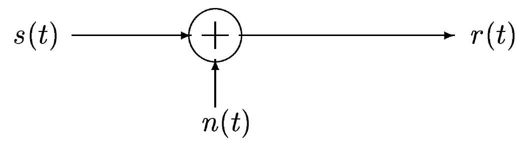
$$s_i(t) = \sum_{j=1}^{N} s_{i,j}\phi_j(t)$$

   Show that

$$\int_0^{T_s} s_m(t)s_n(t)dt = \sum_{j=1}^{N} s_{m,j}s_{n,j}$$

   for any m and n.

3. Consider the waveform channel:



where $n(t)$ is additive white Gaussian noise with power spectral density $\frac{N_0}{2}$, $r(t)$ is the received waveform, and $s(t) = s_i(t)$ when message m_i is to be sent during time $t \in (0, T_s)$. Suppose there are $M = 4$ possible **equally likely** messages and the corresponding signals are:

$$
\begin{aligned}
s_1(t) &= \sqrt{\frac{2}{T_s}} \sin(2\pi f_c t) \\
s_2(t) &= \sqrt{\frac{4}{T_s}} \cos(2\pi f_c t + \frac{\pi}{4}) \\
s_3(t) &= -\sqrt{\frac{2}{T_s}} \sin(2\pi f_c t) \\
s_4(t) &= \sqrt{\frac{4}{T_s}} \cos(2\pi f_c t - \frac{\pi}{4})
\end{aligned}
$$

where $2\pi f_c T_s$ is an integer multiple of 2π. The following identities may be useful for this problem:

$$
\begin{aligned}
\cos(A + B) &= \cos(A)\cos(B) - \sin(A)\sin(B) \\
\cos(A - B) &= \cos(A)\cos(B) + \sin(A)\sin(B)
\end{aligned}
$$

For this entire problem use the following basis functions:

$$
\begin{aligned}
\phi_1(t) &= \sqrt{\frac{2}{T_s}} \sin(2\pi f_c t) \\
\phi_2(t) &= \sqrt{\frac{2}{T_s}} \cos(2\pi f_c t)
\end{aligned}
$$

[8] (a) Find the signal space representation of the signals in this basis.

[7] (b) Draw the complete (optimal) maximum a posteriori (MAP) receiver, with input $r(t)$ and output $\hat{m}$.

[8] (c) Find the symbol error probability in terms of $\frac{E_s}{N_0}$, where E_s is the average symbol energy. (*Hint: If this seems unusually hard, check your answer to (a) carefully.*)

[7] (d) Find the optimal translation of the signal set to minimize the error probability of (c). **How much more energy efficient (in dB) is this system than the system in part (c)?**

4. Suppose we are channel coding a stream of independently and identically distributed bits, each of which is equally likely to be 0 or 1. The bits are grouped in sets of three and input to a linear block channel coder with generator matrix:

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

[12] (a) Find the following:

- The parity check matrix H.
- The minimum distance d_{min}.
- The coset leader corresponding to each syndrome.

[8] (b) Does this code have the largest minimum distance of any $(6, 3)$ linear block code?:

- If you say **"Yes"**: Prove that the d_{min} of this code is the best of any $(6, 3)$ code.
- If you say **"No"**: Find a $(6, 3)$ code with a better minimum distance.

[10] (c) The coded bits out of the encoder of the code are mapped in **pairs** by an amplitude-shift keying (ASK) modulator to the signals whose signal space equivalents s are as follows:

bits	s
00	3
01	1
11	-1
10	-3

The resulting sequence of ASK symbols is sent across a channel that is modeled as a 1-dimensional vector channel $r = s + n$, where n is a zero-mean Gaussian random variable with variance $\frac{N_0}{2}$. Note that there are three transmitted (and thus also three received) ASK symbols for each codeword.

Find the Union Bound to $P(E|\underline{c}_0)$, the probability of a codeword error given the all zeroes codeword was sent, when soft-decision decoding is employed. The solution can be left in units of the vector space (i.e. there is no need to convert to the average symbol energy E_s). (*This is a bit tedious, but important!*)