ECE 314 - Intro to Probability and Random Processes (Spring 2012)

Final Exam

Friday, May 4th, 10:30-12:30pm, Marcus Hall 131

Overview

- The exam consists of four problems for 100 points. The points for each part of each problem are given in brackets - you should spend your **two hours** accordingly.
- The exam is closed book, but you are allowed **three page-sides** of notes. Calculators are **not** allowed. I will provide all necessary blank paper.

Testmanship

- **Full credit will be given only to fully justified answers.**
- Giving the steps along the way to the answer will not only earn full credit but also maximize the partial credit should you stumble or get stuck. If you get stuck, attempt to neatly define your approach to the problem and why you are stuck.
- If part of a problem depends on a previous part that you are unable to solve, explain the method for doing the current part, and, if possible, give the answer in terms of the quantities of the previous part that you are unable to obtain.
- Start each problem on a new page. Not only will this facilitate grading but also make it easier for you to jump back and forth between problems.
- If you get to the end of the problem and realize that your answer must be wrong (e.g. a negative probability, a pmf that does not sum to one), be sure to write “this must be wrong because . . . ” so that I will know you recognized such a fact.
- Academic dishonesty **will** be dealt with **harshly** - the *minimum penalty* will be an “F” for the course.

1. The random variables X and Y have joint pdf given by:

$$f_{X,Y}(x,y) = \begin{cases} kx, & 0 \leq x \leq 1, \quad 0 \leq y \leq x^2 + 1 \\ 0, & \text{else} \end{cases}$$

[4] (a) Sketch the region in the (x, y) plane where $f_{X,Y}(x, y) > 0$.

[4] (b) Find k.

[10] (c) Find $f_X(x)$ and $f_Y(y)$, the marginal probability density functions (pdfs) of X and Y, respectively.

[7] (d) Find $f_{X|Y}(x|y)$, the conditional probability density function of X given Y. For your limits (**which you should not forget**), put y between constant limits, and then give the limits for x.

[5] (e) Suppose you measure $Y = 1.5$ and you want to estimate the value of X. Sketch $f_{X|Y}(x|1.5)$, and then find the x for which $f_{X|Y}(x|1.5)$ is maximized.

2. Suppose random variable X has probability density function (pdf) given by:

$$f_X(x) = \begin{cases} c, & 0 \leq x \leq 2 \\ 0, & \text{else} \end{cases}$$

where c is a constant.

[12] (a) Perform the following (one sentence of justification is fine in each case):

- Find the constant c.
- Find the probability density function (pdf) of $X + 2$. (*Hint: If you understand what $f_X(x)$ is telling you about the probabilities of various outcomes for X, this should be a simple, one-line answer.)*
- Find the probability density function (pdf) of $-X$. (*Hint: This should be a simple, one-line answer.)*
- Find the probability density function (pdf) of $2X$. (*Hint: This should be a simple, one-line answer.)*

Now, suppose the random variable Y has probability density function (pdf) given by:

$$f_Y(y) = \begin{cases} 4 & 0 \leq y \leq \frac{1}{4} \\ 0, & \text{else} \end{cases}$$

and X and Y are **independent.**

[5] (b) Suppose you are trying to obtain a **large** outcome from a given experiment. Would you rather draw your outcome from X (with pdf $f_X(x)$ given at the start of the problem) or Y with pdf $f_Y(y)$? (You can use whatever *reasonable* criterion you want, but be sure to justify your answer).

[8] (c) Find $E[(X - Y)^2]$. (*Hints: (1) Use linearity of expectation. (2) The fractions get a little messy - not too bad.*).

[5] (d) Suppose you apply the following function (a quantizer) $Q(\cdot)$ to X, with pdf $f_X(x)$ given above.

$$Q(x) = \begin{cases} 1/4, & 0 \le x < 1/2 \\ 3/4, & 1/2 \le x < 1 \\ 5/4, & 1 \le x < 3/2 \\ 7/4, & 3/2 \le x \le 2 \\ 0, & \text{else} \end{cases}$$

Let $Z = Q(X)$. Find the probability mass function of Z. (*Hint: This problem is a lot easier than it looks. Start by identifying the possible values of* Z.)

3. Each bit going across a cellular phone link can be received correctly or in error. Suppose that you have two channels: a good channel G and a bad channel B, and that: (1) the channels are encountered with equal probability, and (2) the channel persists for the duration of a given cell phone call. The probability of a bit being in error is 0.1 for channel G and the probability of a bit being in error is 0.5 for channel B, and, given the channel, the error status of any bit is independent of all other bits.

   [10] (a) Suppose you place a very short cell phone call that consists of three bits. Let X be the number of bits in error. Find an expression for $P(X = k)$.

   [5] (b) Suppose that you place a very short cell phone call that consists of three bits. Given that the first two bits are in error, what is the probability that you are operating over a bad channel B?

   [5] (c) Suppose that you place a very short cell phone call that consists of three bits. Given that the first two bits are in error, what is the probability that the third bit is in error?

4. Where needed, you can leave your answers in terms of the $\Phi(\cdot)$ function, which recall is defined as:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du$$

   but is only generally tabulated for $x \ge 0$. In other words, make sure your argument of $\Phi(\cdot)$ is positive.

   [8] (a) I roll a fair six-sided die 500 times. Recall that, for a fair six-sided die, the mean is 3.5 and the variance is approximately 3. Using the Central Limit Theorem, estimate the probability that the sum of the 500 rolls will be ≥ 1800.

   [12] (b) [Assume for this part that you still know the variance of the die roll is 3.] Now I roll an (possibly) unfair die 300 times, and the sum of the rolls comes out to be 900.

   - Give an interval $[a, b]$ in which you are 95% confident the true mean of this (possibly) unfair die falls.
   - Do you think the die is fair or unfair? Be sure to give a short (one-sentence) justification for your answer.