# Final Exam Solutions





1) (a) Let $p_{X,Y}(x,y)$ be given as follows

| | | $y$ | | |
|---|---|---|---|---|
| | | 0 | 1 | 2 |
| $x$ | 0 | 1/8 | 1/4 | 0 |
| | 1 | 1/8 | 1/4 | 1/4 |

$$H(X \mid Y=2) = 0$$
$$H(X \mid Y=0) = 1$$

$$H(X) = \mathcal{H}_2(3/8) = -3/8 \log_2 3/8 - 5/8 \log_2 5/8$$
$$= 0.95$$

(b) Let
$$p_Y(y) = \begin{cases} 1/2 & y=-1 \\ 1/2 & y=1 \end{cases}$$

and $X = Y^2$

$$I(X;Y) = H(X) - H(X \mid Y)$$
$$= H(X) = 0$$

$$H(Y) = 1$$

(c)

$$H(Z) \doteq H(X, Z)$$
$$= H(X) + H(Z|X) \quad \text{chain rule}$$
$$= H(X) + H(Y|Z) \quad Z \text{ and } Y \text{ are 1:1 given } X.$$
$$\leq H(X) + H(Y)$$

First Example: $(H(Z) = H(X) + H(Y))$

Pick $p_X(x) = \begin{cases} 1/2 & x = 0 \\ 1/2 & x = 2 \end{cases}$

$$p_Y(y) = \begin{cases} 1/2 & y = 0 \\ 1/2 & y = 1 \end{cases}$$

and let $X, Y$ be independent.

Then $H(Z) = 2 = 1 + 1 = H(X) + H(Y)$

Second Example: $(H(Z) < H(X) + H(Y))$

Pick $p_X(x) = \begin{cases} 1/2 & x = 0 \\ 1/2 & x = 1 \end{cases}$ and $p_Y(y) = \begin{cases} 1/2 & y = -1 \\ 1/2 & y = 0 \end{cases}$

and let $X, Y$ be independent.

Then $H(Z) = 3/2 < 2 = H(X) + H(Y)$

2) $I(X;Y) = H(Y) - H(Y|X)$

Let $p = P(X=0)$

$$p_Y(y) = \begin{cases} 1/2 + \frac{p}{2} & y=0 \\ 1/2 - p/2 & y=1 \end{cases} \Rightarrow H(Y) = \mathcal{H}_2\left(\frac{1+p}{2}\right)$$

$$I(X;Y) = \mathcal{H}_2\left(\frac{1+p}{2}\right) - \left(p\, \overset{0}{\cancel{H(Y|X=0)}} + (1-p)\, \overset{1}{\cancel{H(Y|X=1)}}\right)$$

$$= \mathcal{H}_2\left(\frac{1+p}{2}\right) - 1 + p$$

$$\frac{d}{dp} I(X;Y) = \log_2\left(\frac{1 - \frac{(1+p)}{2}}{\frac{(1+p)}{2}}\right)\left(+\frac{1}{2}\right) + 1$$

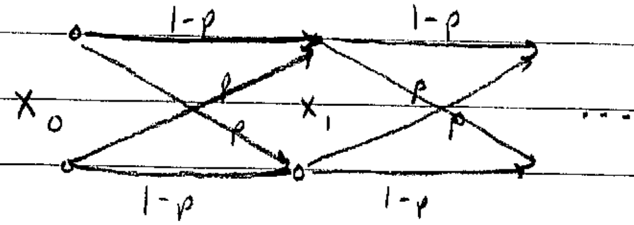
$$= \log_2\left(\frac{(1-p)}{(1+p)}\right)\left(+1/2\right) + 1$$

$$\Rightarrow \log_2 \frac{(1-p)}{(1+p)} = -2 \Rightarrow p = 3/5$$

$$\Rightarrow C = \mathcal{H}_2(4/5) - 2/5$$

3)

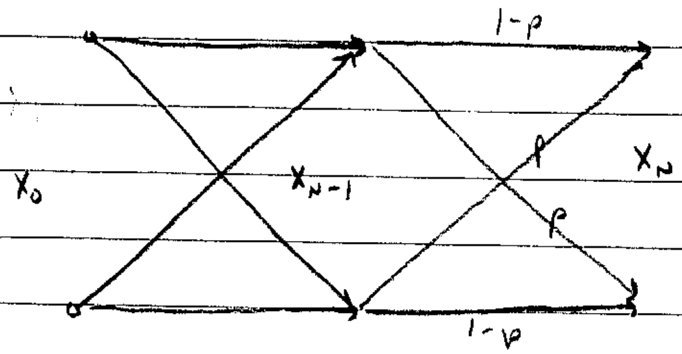
(a)-(b)




By induction (obviously true for $N=1$)
Suppose that the cascade of $N-1$ channels is a BSC with crossover probability

$$\frac{1}{2}\left(1-(1-2p)^{N-1}\right)$$

Then




$$P(X_N=0\,|\,X_0=1) = P(X_N=0\,|\,X_{N-1}=0)\,P(X_{N-1}=0\,|\,X_0=1) + P(X_N=0\,|\,X_{N-1}=1)\,P(X_{N-1}=1\,|\,X_0=1)$$

$$= (1-p)\,\frac{1}{2}\left(1-(1-2p)^{N-1}\right) + p\left(1-\frac{1}{2}\left(1-(1-2p)^{N-1}\right)\right)$$

$$= \frac{1}{2} - \left(\frac{(1-p)}{2}-\frac{p}{2}\right)(1-2p)^{N-1}$$

$$= \frac{1}{2}\left(1-(1-2p)^{N}\right)$$

By symmetry, $P(X_N = 1 \mid X_0 = 0)$ is the same.

Let $p_N = \frac{1}{2}\left(1 - (1-2p)^N\right)$

$$I(X_0; X_N) = H(X_n) - \mathcal{H}_2(p_N)$$

$$\leq 1 - \mathcal{H}_2(p_N)$$

$$\lim_{N \to \infty} I(X_0; X_N) \leq \lim_{N \to \infty} 1 - \mathcal{H}_2(p_N)$$

$$= 1 - \mathcal{H}_2\left(\lim_{N \to \infty} p_N\right) \quad \text{(continuity of } \mathcal{H}_2(\cdot)\text{)}$$

$$= 1 - \mathcal{H}_2(1/2) \quad (p \neq 0 \text{ nor } 1)$$

$$= 0$$

$$\Rightarrow \lim_{N \to \infty} C_N = 0.$$

3)

(c) $1 - \mathcal{H}_2(p)$.

4)

$$E[X_1^2 + X_2^2] \le 2P \qquad \sigma_1^2 > \sigma_2^2$$

Simply apply Cover, 10.75,

$$(v - \sigma_1^2)^+ + (v - \sigma_2^2)^+ = 2P$$

The question asks at what $P$ does the solution $v$ exceed $\sigma_1^2$. For $v < \sigma_1^2$,

$$v - \sigma_2^2 = 2P \Rightarrow v = \sigma_2^2 + 2P$$

This solution hits $\sigma_1^2$ at:

$$\sigma_1^2 = \sigma_2^2 + 2P$$

$$\Rightarrow P = \frac{\sigma_1^2 - \sigma_2^2}{2}$$

5)

[Problem not graded because:

- nobody put much effort into it
- nobody came close.]

It is not a DMC. You need to re-establish achievability.

6)

$H(X) = \mathcal{H}_2(p)$

$Y = X \oplus Z$ $\qquad P(Y=0) = P(X=0)\,P(Z=0) + P(X=1)\,P(Z=1)$

$Y \oplus Z = X \oplus Z \oplus Z$ $\qquad = (1-p)(1-r) + rp$

$\Rightarrow X = Y \oplus Z$ $\qquad = 1 - p - r + 2rp$

$P(Y=1) = p + r - 2rp$

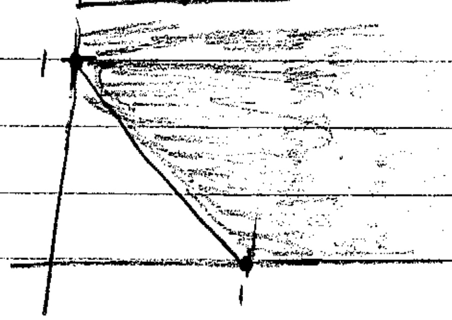
$\Rightarrow H(Y) = \mathcal{H}_2(p + r - 2rp)$

$H(X|Y) = H(Y|X) = \mathcal{H}_2(r)$

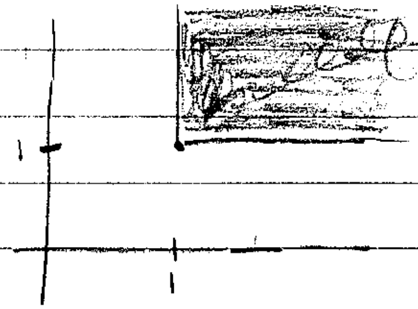
$H(X,Y) = H(X) + H(Y|X)$

$= \mathcal{H}_2(p) + \mathcal{H}_2(r)$

$p = 1/2,\ r = 0$




$p = 1/2,\ r = 1/2$



7) Divide all messages randomly over $2^{NR_0}$ bins. Give assignment to $S_2$ and D. For a given message, $S_2$ sends index of its bin ($NR_0$ bits)

$$P_E = P\left(\bar{E}_0 \cup \left(\bigcup_{i=1}^{2^{NR}-1} (E_i \cap C_i)\right)\right)$$ $i^{th}$ codeword in bin of xmitted codeword

$$\leq P(\bar{E}_0) + \sum_{i=1}^{2^{NR}-1} P(E_i \cap C_i)$$

$$= P(\bar{E}_0) + \sum_{i=1}^{2^{NR}-1} P(E_i)\, P(C_i)$$

$$= P(\bar{E}_0) + \sum_{i=1}^{2^{NR}-1} 2^{-N(I(X;Y)-3\epsilon)}\, 2^{-NR_0}$$

$$\leq \delta/2 + 2^{N(R - I(X;Y) - R_0 - 3\epsilon)}$$

Since $\epsilon$ is arbitrary, just need

$$R_0 > R - I(X;Y)$$

and thus

$$C_2 > R - I(X;Y).$$