ECE 603 - Probability and Random Processes, Fall 2016

Final Exam

December 21, 8am-10am

ELAB 304

Overview

- The exam consists of four problems for 115 points. The points for each part of each problem are given in brackets - you should spend your **two hours** accordingly.
- The exam is closed book, but you are allowed **three page-sides** of notes. Calculators are not allowed. I will provide all necessary blank paper.

Testmanship

- **Full credit will be given only to fully justified answers.**
- Giving the steps along the way to the answer will not only earn full credit but also maximize the partial credit should you stumble or get stuck. If you get stuck, attempt to neatly define your approach to the problem and why you are stuck.
- If part of a problem depends on a previous part that you are unable to solve, explain the method for doing the current part, and, if possible, give the answer in terms of the quantities of the previous part that you are unable to obtain.
- Start each problem on a new page. Not only will this facilitate grading but also make it easier for you to jump back and forth between problems.
- If you get to the end of the problem and realize that your answer must be wrong, be sure to write "this must be wrong because ..." so that I will know you recognized such a fact.
- Academic dishonesty will be dealt with harshly - the *minimum penalty* will be an "F" for the course.

1. I form the sequence $X_1, X_2, \ldots$, as follows. I flip a fair coin (probability of "Heads" = 1/2; probability of "Tails" = 1/2). **Note that the coin is flipped only once.** If the coin is "Heads", I draw the sequence $X_1, X_2, \ldots$ by *independent* draws from a probability density function:

$$f_X^{(H)}(x) = \begin{cases} 1/10, & 0 \leq x \leq 10 \\ 0, & \text{else} \end{cases}$$

If the coin is "Tails", I draw the sequence $X_1, X_2, \ldots$ by *independent* draws from a probability density function:

$$f_X^{(T)}(x) = \begin{cases} 1/20, & 0 \leq x \leq 20 \\ 0, & \text{else} \end{cases}$$

Consider the resulting sequence $X_1, X_2, \ldots$. **We call X_i "big" if $X_i > 5$.**

[5] (a) Find the probability that X_1 is "big".

[8] (b) We observe that X_1 is "big". Given this information, what is the probability that X_2 is "big"?

[7] (c) Write an exact expression for the probability that exactly 15 of the variables in $\{X_1, X_2, \ldots, X_{20}\}$ are "big".

[5] (d) We observe the sequence and note that 15 of the variables in $\{X_1, X_2, \ldots, X_{20}\}$ are "big". What is the probability that the coin flip was "Tails"?

[5] (e) Suppose that I observe $X_1, X_2, \ldots, X_{1000}$. Estimate the probability that more than 750 of the X_i are "big"? (There is one good answer, but there are multiple ways to justify such.)

2. [15] I have a stick of length 10 meters. I lay the stick out left-to-right in front of me. The stick is then broken at two points, B_1 meters and B_2 meters from the left end. Suppose B_1 and B_2 are random variables chosen independently and uniformly from the (uncountable) interval $[0, 10]$ meters; that is, B_1 and B_2 are independent and

$$f_{B_1}(x) = f_{B_2}(x) = \begin{cases} 1/10, & 0 \leq x \leq 10 \\ 0, & \text{else} \end{cases}$$

After the stick is broken in these two locations, you have three pieces of a stick.

Let X be the length of the shortest piece of the stick after it is broken. Find $f_X(x)$, the probability density function of X.

3. The random variables X and Y have joint probability density function (pdf) given by:

$$f_{X,Y}(x,y) = \begin{cases} c\, e^{-(\lambda x + \mu y)}, & x \geq 0, y \geq 0 \\ 0, & \text{else} \end{cases}$$

for constants $\mu > 0$, $\lambda > 0$.

[5] (a) Find c in terms of μ and λ.

[7] (b) Find the marginal probability density functions $f_X(x)$ and $f_Y(y)$ of X and Y, respectively.

[5] (c) Find $f_{Y|X}(y|x)$, the conditional probability density function of Y given X. For your limits (**which you should not forget**), put x between constant limits (**not dependent on** y), and then give the limits for y.

[5] (d) Are X and Y independent?

[8] (e) Find $P(Y > X)$, the probability that $Y > X$ (in terms of λ and μ).

4. Consider the Gaussian random process $X(t)$ for $t \geq 0$ with mean $m_X(t) = E[X(t)] = 0$ and autocorrelation function $R_X(t_1, t_2) = E[X(t_1)X(t_2)] = 4\,\min(t_1, t_2)$, where $\min(x, y)$ is the minimum of x and y.

[5] (a) Is this process wide-sense stationary (WSS)? Is it strict-sense stationary (SSS)? **(Don't forget the SSS part!)**

[5] (b) If I define the power at time t as $\mathcal{P}(t) = E[X^2(t)]$, find $\mathcal{P}(t)$.

[5] (c) Consider the sample $X(2)$. What is the probability $P(X(2) > 1)$?

[10] (d) Suppose I form the discrete-time random process (i.e. random sequence) $V_1, V_2, \ldots$, by sampling $X(t)$ in the following way (which is somewhat unusual; note that successive sample times get closer in time to $t = 0$ as n gets larger): $V_n = X(1/n)$. Does V_n converge? If so, to what and in what ways? Consider convergence in mean square, probability, and distribution? (You can use one form of convergence to imply another.)

[10] (e) I sample the random process $X(t)$ once per second by forming $Y_n = X(n)$, $n = 0, 1, 2, \ldots$, and then form $Z_n = Y_n - Y_{n-1}$, for $n = 1, 2, \ldots$. Find the mean function $m_Z[n] = E[Z_n]$ and autocorrelation function $R_Z[m, n] = E[Z_m Z_n]$ for all $m > 0$ and $n > 0$. (Simplify your answers as much as possible for full credit; this will also help you with the next part.)

[5] (f) For Z_n as defined in the previous part, is Z_n wide-sense stationary?