ECE 603 - Probability and Random Processes, Fall 2015

Final Exam

December 18th, 8:00-10:00am

ELAB 323

Overview

- The exam consists of four problems for 125 points. The points for each part of each problem are given in brackets - you should spend your **two hours** accordingly.
- The exam is closed book, but you are allowed **two page-sides** of notes. Calculators are not allowed. I will provide all necessary blank paper.

Testmanship

- **Full credit will be given only to fully justified answers.**
- Giving the steps along the way to the answer will not only earn full credit but also maximize the partial credit should you stumble or get stuck. If you get stuck, attempt to neatly define your approach to the problem and why you are stuck.
- If part of a problem depends on a previous part that you are unable to solve, explain the method for doing the current part, and, if possible, give the answer in terms of the quantities of the previous part that you are unable to obtain.
- Start each problem on a new page. Not only will this facilitate grading but also make it easier for you to jump back and forth between problems.
- If you get to the end of the problem and realize that your answer must be wrong, be sure to write "this must be wrong because ..." so that I will know you recognized such a fact.
- Academic dishonesty will be dealt with harshly - the *minimum penalty* will be an "F" for the course.

1. The random variables X and Y have joint probability density function (pdf) given by:

$$f_{X,Y}(x,y) = \begin{cases} cy, & 0 \le y \le 1, \quad y^2 \le x \le 1, \\ 0, & \text{else} \end{cases}$$

[5] (a) Find c.

[5] (b) Write an integral expression for $P(Y > 1 - X)$. You do not need to evaluate the integral, but you need to specify all quantities (integrand, integral limits) precisely!

[5] (c) Find $f_X(x)$, the marginal probability density function (pdf) of X.

[5] (d) Find $f_{Y|X}(y|x)$, the conditional probability density function of Y given X. For your limits (**which you should not forget**), put x between constant limits (**not dependent on** y), and then give the limits for y.

[5] (e) Are X and Y independent?

[5] (f) Suppose you measure $X = 0.5$ and you want to estimate the value of Y. Select a "reasonable" value of Y for your estimate (and justify).

2. Suppose a soccer player Alice, who either scores 1 goal in a game or scores 0 goals in a game (i.e. she never scores more than one goal), plays in a series of three soccer games against a rival team. For the first game, she has a 0.6 probability of scoring a goal. For the second and third games: given whether Alice scored in the previous game, the event of Alice scoring a goal is independent of games before the previous game (i.e. if G_i is the event she scores in game i, $P(G_3|G_1, G_2) = P(G_3|G_2)$). If she scored in game $n-1$, she has a 0.8 probability of scoring in game n; if she did not score in game $n-1$, she has a 0.5 probability of scoring in game n.

[5] (a) What is the probability that Alice scores a goal in the second game?

[5] (b) Given that she scores a goal in the second game, what is the probability that she scored a goal in the first game.

[10] (c) Given that she score a goal in the first game, what is the probability that she scores a goal in the third game?

3. Let $X_0, X_1, X_2, \ldots$ be a discrete-time random process consisting of **independent** random variables, **each** with cumulative distribution function:

$$F_X(x) = \begin{cases} 0, & x \leq -1 \\ 1 - x^2, & -1 \leq x \leq 0 \\ 1, & x \geq 0 \end{cases}$$

[5] (a) Find $P(X_0 \geq -\frac{1}{2})$.

[5] (b) Find $E[X_0]$, the expected value of X_0.

[5] (c) Find $Var[X_0]$, the variance of X_0.

[10] (d) Let $U = \sum_{i=1}^{180} X_i$. Estimate $P(U \geq -115)$.

Define a new random process $Y_1, Y_2, Y_3, \ldots$ such that: $Y_i = X_i - X_{i-1}$; that is, $Y_1 = X_1 - X_0$, $Y_2 = X_2 - X_1$, $Y_3 = X_3 - X_2, \ldots$.

[5] (e) Find $m_Y[n] = E[Y_n]$, the mean function of Y_n.

[10] (f) Find $R_Y[m, n] = E[Y_m Y_n]$, the autocorrelation function of Y_n. Is Y_n wide-sense stationary?

4. Let $X(t)$ be a stationary Gaussian random process with mean zero and autocorrelation function $R_X(\tau) = 20\text{sinc}^2(80\tau)$. Let $Y(t) = 4\cos(2\pi 30t) + X(t)$.

[5] (a) Find the power $E[X^2(t)]$ in $X(t)$.

[10] (b) For $Y(t)$:

- Find the mean function $m_Y(t) = E[Y(t)]$.
- Find the autocorrelation function $R_Y(t_1, t_2) = E[Y(t_1)Y(t_2)]$.
- Is $Y(t)$ wide-sense stationary?

[8] (c) Is $Y(t)$ a Gaussian random process?

[12] (d) Suppose that $Y(t)$ is run through a filter with impulse response $h(t) = 40\text{sinc}(80t)$ to yield an output $Z(t)$. Define the average power at the output of the filter as:

$$P_Z = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} E[Z^2(t)]dt$$

Find P_Z.