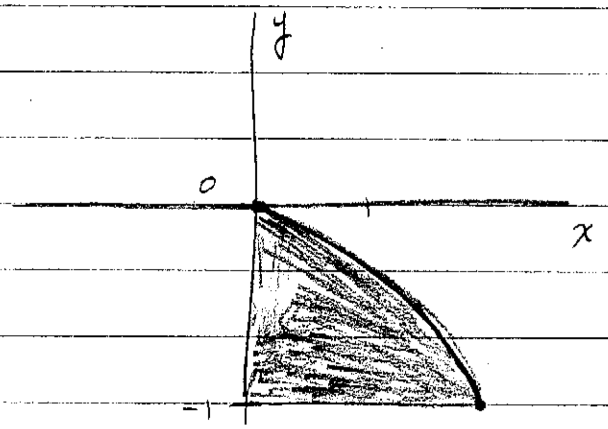
Final Exam Solutions

1)

(a)



(b)

$$\int_{-1}^{0}\int_{0}^{\sqrt{-y}} c\,x\,dx\,dy = \int_{-1}^{0}\left(c\,x^2/2\right)\Big|_0^{\sqrt{-y}} dy = \int_{-1}^{0} -cy/2\,dy$$

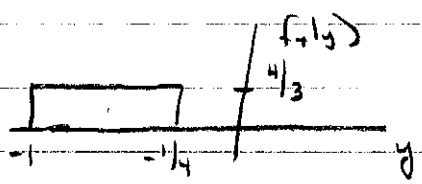
$$= -cy^2/4\Big|_{-1}^{0} = c/4 \Rightarrow c = 4$$

(c) $$\underset{0\le x\le 1}{f_X(x)} = \int_{-1}^{-x^2} 4x\,dy = 4xy\Big|_{-1}^{-x^2} = -4x^3 + 4x$$

$$\Rightarrow f_X(x) = \begin{cases} 4x - 4x^3, & 0 \le x \le 1 \\ 0, & \text{else} \end{cases}$$ ← integrates to 1! ✓

(d) $$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \begin{cases} \dfrac{\cancel{4x}}{\cancel{4x}-\cancel{4x}^3} = \dfrac{1}{1-x^2}, & 0 \le x \le 1,\ -1 \le y \le -x^2 \\ 0, & \text{else} \end{cases}$$

(e) Look at $$f_{Y|X}(y|0.5) = \begin{cases} 4/3, & -1 \le y \le -1/4 \\ 0, & \text{else} \end{cases}$$

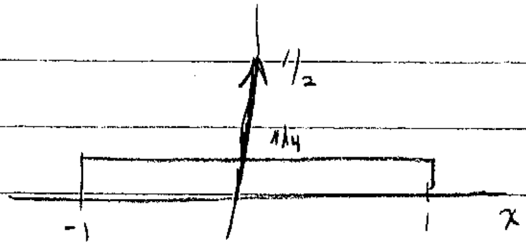


All equally likely
I would pick average:
$E[Y|X=1/2] = -5/8$

2) (a)

$\Omega = [-1, 1] \quad \mathcal{A} = \mathbb{B}$

Now, $f_\omega(x)$ looks like



$$\Rightarrow P((a,b)) = \begin{cases} \frac{1}{4}(b-a), & -1 \le a < b \le 0 \\ \frac{1}{4}(b-a) + \frac{1}{2}, & -1 \le a < 0 < b \le 1 \\ \frac{1}{4}(b-a) & 0 \le a < b \le 1 \\ 0, & \text{else} \end{cases}$$

(b) • $P(\omega^2 + 1 < 3/2) = P(\omega^2 < 1/2) = P(|\omega| < 1/\sqrt{2}) = P((-1/\sqrt{2}, 1/\sqrt{2}))$

$= 1/4(1/\sqrt{2} - (-1/\sqrt{2})) + 1/2 = 1/2 + 1/2\sqrt{2}$

• $P(\{\omega = 0\} | \{\omega^2 + 1 < 3/2\}) = \dfrac{P(\{\omega = 0\} \cap \{\omega^2 + 1 < 3/2\})}{P(\omega^2 + 1 < 3/2)} = \dfrac{1/2}{1/2 + 1/2\sqrt{2}} = \dfrac{1}{1 + 1/\sqrt{2}}$

(c) I claim $Z_n \to 1$ in all ways:

a.s. : $1 + \omega^n \to 1$ except at -1 and 1

but $P((-1, 1)) = 1$ ✓

m.s. $E[((1+\omega^n) - 1)^2] = E[\omega^{2n}] = 0^{2n} \cdot 1/2 + \int_{-1}^{1} \omega^{2n} \cdot 1/4 \, d\omega$

$= \frac{1}{4} \frac{1}{2n+1} \omega^{2n+1} \Big|_{-1}^{1} = \frac{2}{4(2n+1)} \to 0$ as $n \to \infty$ ✓

$Z_n \xrightarrow{m.s.} 1 \Rightarrow Z_n \xrightarrow{P} 1 \Rightarrow Z_n \xrightarrow{D} 1$

3) (a)

$$E[Y] = E\left[\sum_{i=1}^{N} X_i\right] \underset{\substack{\uparrow \\ \text{linearity of} \\ \text{Expectation}}}{=} \sum_{i=1}^{N} E[X_i] = N\mu$$

(b)
$$E[Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f_{Y|N}(y|n)\, f_N(n)\, dy\, dn$$

$$= \int_{-\infty}^{\infty} \underbrace{\int_{-\infty}^{\infty} y\, f_{Y|N}(y|n)\, dy}_{E[Y\,|\,N=n]}\, f_N(n)\, dn$$

$$= \int_{-\infty}^{\infty} N\mu\, f_N(n)\, dn$$

$$= \mu \int_{-\infty}^{\infty} N\, f_N(n)\, dn$$

$$= N\mu$$

4) (a)

$$E[m(i)m(j)] = \delta_{i,j}$$

$\Rightarrow m(i), m(j)$ independent, all $i \neq j$

By SLLN $Z_N \xrightarrow{a.s.} E[m(i)] = 0 \Rightarrow Z_n \xrightarrow{P} 0 \Rightarrow Z_n \xrightarrow{D} 0$

Also $E[(Z_n - 0)^2] = \frac{1}{N^2} E\left[\left(\sum_{i=1}^{N} m(i)\right)^2\right]$

$$= \frac{1}{N^2} N \cdot R_m(0) \to 0 \Rightarrow Z_n \xrightarrow{ms} 0$$

(b)

$$Z_3 = m(0) + m(1/2) + m(1)$$

$m(t)$ is a Gaussian RP

$\Rightarrow m(0), m(1/2), m(1)$ are jointly Gaussian

$\Rightarrow Z_3$ is Gaussian

$$E[Z_3] = \tfrac{1}{3}\left(\cancel{E[m(0)]}^{0} + \cancel{E[m(1/2)]}^{0} + \cancel{E[m(1)]}^{0}\right) = 0$$

linearity of $E[\cdot]$.

$$\mathrm{Var}[Z_3] = E[Z_3^2] = \tfrac{1}{9} E\left[(m(0) + m(1/2) + m(1))^2\right]$$

$$= \tfrac{1}{9}\Big(\cancel{E[m^2(0)]}^{1} + \cancel{E[m^2(1/2)]}^{1} + \cancel{E[m^2(1)]}^{1}$$

$$+ 2\,\cancel{E[m(0)m(1/2)]}^{1/2} + 2\,\cancel{E[m(1/2)m(1)]}^{1/2}$$

$$+ 2\,\cancel{E[m(0)m(1)]}^{0}\Big) = 5/9$$

$$f_{Z_3}(x) = \frac{1}{\sqrt{2\pi 5/9}} e^{-x^2/10/9}$$

(c)

I claim $Z_n \xrightarrow{m.s.} 0$

$$E[(Z_n - 0)^2] = E\left[\left(\frac{1}{N}\sum_{i=1}^{N} m(i|_2)\right)^2\right]$$

$$= \frac{1}{N^2} E\left[\sum_{i=1}^{N}\sum_{j=1}^{N} m(i|_2)\, m(j|_2)\right]$$

$$= \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} \underbrace{E[m(i|_2)\, m(j|_2)]}_{= \begin{cases} 1, & i=j \\ 1/2, & |i-j|=1 \\ 0, & \text{else} \end{cases}}$$

$$= \frac{1}{N^2}\left(N + 2\cdot(N-1)\cdot 1/2\right)$$

$\to 0$ as $N \to \infty$,

$$Z_n \xrightarrow{m.s.} 0 \Rightarrow Z_n \xrightarrow{P} 0 \Rightarrow Z_n \xrightarrow{D} 0$$

5) (a)

$$H_2(f) = 2\, H_1(f)$$

$$\Rightarrow |H_2(f)|^2 = 4|H_1(f)|^2$$

$$\Rightarrow P_2 = 4 \cdot 3 = 12 \text{ milliwatts}$$

(b)

$$H_3(f) = e^{-j2\pi f} H_1(f)$$

$$\Rightarrow |H_3(f)|^2 = |H_1(f)|^2$$

$$\Rightarrow P_3 = 1 \cdot 3 = 3 \text{ milliwatts}$$