ECE 603 - Probability and Random Processes, Fall 2010

Final Exam

December 14th, 10:30am-12:30pm, Marston 132

Overview

- The exam consists of five problems for 110 points. The points for each part of each problem are given in brackets - you should spend your **two hours** accordingly.
- The exam is closed book, but you are allowed **three page-sides** of notes. Calculators are not allowed. I will provide all necessary blank paper.

Testmanship

- **Full credit will be given only to fully justified answers.**
- Giving the steps along the way to the answer will not only earn full credit but also maximize the partial credit should you stumble or get stuck. If you get stuck, attempt to neatly define your approach to the problem and why you are stuck.
- If part of a problem depends on a previous part that you are unable to solve, explain the method for doing the current part, and, if possible, give the answer in terms of the quantities of the previous part that you are unable to obtain.
- Start each problem on a new page. Not only will this facilitate grading but also make it easier for you to jump back and forth between problems.
- If you get to the end of the problem and realize that your answer must be wrong, be sure to write "this must be wrong because ..." so that I will know you recognized such a fact.
- Academic dishonesty will be dealt with harshly - the *minimum penalty* will be an "F" for the course.

Some potentially useful information

$$\begin{aligned}\cos(\theta) &= \tfrac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)\\ \sin(\theta) &= \tfrac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)\end{aligned}$$

$$\begin{aligned}\sin(a \pm b) &= \sin(a)\cos(b) \pm \cos(a)\sin(b)\\ \cos(a \pm b) &= \cos(a)\cos(b) \mp \sin(a)\sin(b)\end{aligned}$$

$$\begin{aligned}\cos(a)\cos(b) &= \tfrac{1}{2}\left[\cos(a-b) + \cos(a+b)\right]\\ \sin(a)\sin(b) &= \tfrac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]\\ \sin(a)\cos(b) &= \tfrac{1}{2}\left[\sin(a-b) + \sin(a+b)\right]\end{aligned}$$

Time Function	**Fourier Transform**
$x(at+b)$	$\frac{1}{\lvert a\rvert}X\left(\frac{f}{a}\right)e^{j2\pi\frac{b}{a}f}$
$p(t) = \begin{cases} 1 & \lvert t\rvert \le 1/2 \\ 0 & \lvert t\rvert > 1/2 \end{cases}$	$P(f) = \frac{\sin(\pi f)}{\pi f} = \mathrm{sinc}(f)$
$\cos(2\pi f_0 t)$	$\frac{1}{2}\left[\delta(f-f_0) + \delta(f+f_0)\right]$
$\sin(2\pi f_0 t)$	$\frac{1}{2j}\delta(f-f_0) - \frac{1}{2j}\delta(f+f_0)$
$e^{-a\lvert t\rvert}, \quad a > 0$	$\frac{2a}{a^2+(2\pi f)^2}$
$x(t) = \begin{cases} 1-\lvert t\rvert & \lvert t\rvert \le 1 \\ 0 & \lvert t\rvert > 1 \end{cases}$	$X(f) = \mathrm{sinc}^2(f)$

Parseval's Relation: If $X(f)$ is the Fourier Transform of $x(t)$,

$$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt = \int_{-\infty}^{\infty} \lvert X(f)\rvert^2 df$$

1. The random variables X and Y have joint probability density function:

$$f_{X,Y}(x,y) = \begin{cases} cx, & -1 \leq y \leq 0, 0 \leq x \leq \sqrt{-y} \\ 0, & \text{otherwise} \end{cases}$$

[5] (a) Shade the region in the (x, y) plane where $f_{X,Y}(x, y)$ is non-zero.

[5] (b) Find c.

[5] (c) Find $f_X(x)$, the marginal probability density function of X.

[5] (d) Find $f_{Y|X}(y|x)$, the conditional probability density function of Y given X. **For this, write your limits for x between constants and your limits on y as (potentially) functions of x.**

2. Consider the following experiment: I flip a fair coin. If the coin comes up "heads", I record the outcome 0 (zero) in my notebook; if the coin comes up "tails", I generate a random number from a uniform distribution on the interval $[-1, 1]$, and record the result as the outcome in my notebook. As has been the standard from class, denote the outcome of the experiment ω.

[10] (a) Find a non-trivial space $(\Omega, \mathcal{A}, P)$ for the experiment. (*If you have a hard time finding P, give the probability density function of ω for partial credit.*)

[10] (b) Let $Y = \omega^2 + 1$.

- Find $P(Y < \frac{3}{2})$.
- Given $\{Y < \frac{3}{2}\}$, find the probability that the coin came up "heads".

[10] (c) Let $Z_n(\omega) = 1 + \omega^n$. Determine whether the sequence $\{Z_n\}$ converges, and, if so, to what and in what ways? Consider almost sure convergence, mean square convergence, convergence in probability, and convergence in distribution. You can do them in any order that you want, and you can use one type of convergence to imply another when appropriate.

3. For a fixed N, and independent and identically distributed sequence of X_i's, each with mean $\mu = E[X_i]$ and variance σ^2, we know that, if $Y = \sum_{i=1}^{N} X_i$, then:

$$E[Y] = E[\sum_{i=1}^{N} X_i] = \sum_{i=1}^{N} E[X_i] = NE[X_i] = N\mu$$

Answer each of the parts below independently.

[5] (a) Suppose that the X_i are still identically distributed with $\mu = E[X_i]$ and variance σ^2, but now they are correlated such that $cov(X_i, X_j) = \sigma^2 \rho^{i-j}$, for some $0 < \rho < 1$. Again, let $Y = \sum_{i=1}^{N} X_i$. Find $E[Y]$ in terms of N, μ, σ^2, and ρ.

[10] (b) Now, go back to assuming an independent and identically distributed sequence of X_i's, each with mean $\mu = E[X_i]$ and variance σ^2. But now suppose that N is also random. That is, suppose

I have a non-negative discrete random variable N; I generate N and then sum together the first N elements of the sequence $\{X_i\}$. Call the result Y. Find the expected value of Y in terms of $E[N]$, the variance σ_N^2 of N, μ and σ^2. (*Hint: Write the joint probability density function of Y and N as $f_{Y|N}(y|n)f_N(n)$, and use the definition of the expectation of Y to get started. You can get to the answer in just a few steps if you head the right direction.*)

4. Consider a stationary Gaussian random process $M(t)$ with mean zero and autocorrelation function

$$R_M(\tau)\begin{cases} 1-|\tau|, & |\tau|\leq 1 \\ 0, & |\tau|>1 \end{cases}$$

[10] (a) Consider the sequence $Z_N = \frac{1}{N}\sum_{i=1}^{N} M(i)$ formed by averaging samples of $M(t)$ taken one second apart. Determine whether the sequence $\{Z_N\}$ converges, and, if so, to what and in what ways? Consider almost sure convergence, mean square convergence, convergence in probability, and convergence in distribution. You can do them in any order that you want, and you can use one type of convergence to imply another when appropriate.

[20] (b) Consider the sequence $Z_N = \frac{1}{N}\sum_{i=1}^{N} M(i/2)$ formed by averaging samples of $M(t)$ taken one-half second apart.

- Find the probability density function of Z_3.
- (*Note: Starting from the probability density function of Z_3 is probably* **not** *the easiest way to solve this part.*) Determine whether the sequence $\{Z_N\}$ converges, and, if so, to what and in what ways? Consider **only** mean square convergence, convergence in probability, and convergence in distribution. You can do them in any order that you want, and you can use one type of convergence to imply another when appropriate.

5. Suppose I have a zero-mean stationary Gaussian random process $M(t)$ with autocorrelation function $\mathrm{sinc}(\tau)$. Suppose $M(t)$ is input to a linear time-invariant filter with impulse response $h_1(t) = c\cdot\mathrm{sinc}^2(t)$. You measure the power at the output of the filter to be 3 milliWatts.

[7] (a) Suppose $M(t)$ is input to a linear time-invariant filter with impulse response $h_2(t) = 2c\cdot\mathrm{sinc}^2(t)$. Find the power at the output of the filter.

[8] (b) Suppose $M(t)$ is input to a linear time-invariant filter with impulse response $h_3(t) = c\cdot\mathrm{sinc}^2(t-1)$. Find the power at the output of the filter.