ECE 603 - Probability and Random Processes, Fall 2009

Final Exam

December 15th, 10:30am-12:30pm, LGRT 0321

Overview

- The exam consists of five problems for 110 points. The points for each part of each problem are given in brackets - you should spend your **two hours** accordingly.
- The exam is closed book, but you are allowed **three page-sides** of notes. Calculators are not allowed. I will provide all necessary blank paper.

Testmanship

- **Full credit will be given only to fully justified answers.**
- Giving the steps along the way to the answer will not only earn full credit but also maximize the partial credit should you stumble or get stuck. If you get stuck, attempt to neatly define your approach to the problem and why you are stuck.
- If part of a problem depends on a previous part that you are unable to solve, explain the method for doing the current part, and, if possible, give the answer in terms of the quantities of the previous part that you are unable to obtain.
- Start each problem on a new page. Not only will this facilitate grading but also make it easier for you to jump back and forth between problems.
- If you get to the end of the problem and realize that your answer must be wrong, be sure to write "this must be wrong because ..." so that I will know you recognized such a fact.
- Academic dishonesty will be dealt with harshly - the *minimum penalty* will be an "F" for the course.

<u>Some potentially useful information</u>

$$\begin{aligned}\cos(\theta) &= \tfrac{1}{2}\left(e^{j\theta}+e^{-j\theta}\right)\\ \sin(\theta) &= \tfrac{1}{2j}\left(e^{j\theta}-e^{-j\theta}\right)\end{aligned}$$

$$\begin{aligned}\sin(a\pm b) &= \sin(a)\cos(b)\pm\cos(a)\sin(b)\\ \cos(a\pm b) &= \cos(a)\cos(b)\mp\sin(a)\sin(b)\end{aligned}$$

$$\begin{aligned}\cos(a)\cos(b) &= \tfrac{1}{2}\left[\cos(a-b)+\cos(a+b)\right]\\ \sin(a)\sin(b) &= \tfrac{1}{2}\left[\cos(a-b)-\cos(a+b)\right]\\ \sin(a)\cos(b) &= \tfrac{1}{2}\left[\sin(a-b)+\sin(a+b)\right]\end{aligned}$$

Time Function	**Fourier Transform**
$x(at+b)$	$\frac{1}{\lvert a\rvert}X\left(\frac{f}{a}\right)e^{j2\pi\frac{b}{a}f}$
$p(t)=\begin{cases}1 & \lvert t\rvert\le 1/2\\ 0 & \lvert t\rvert > 1/2\end{cases}$	$P(f)=\frac{\sin(\pi f)}{\pi f}=\text{sinc}(f)$
$\cos(2\pi f_0 t)$	$\frac{1}{2}\left[\delta(f-f_0)+\delta(f+f_0)\right]$
$\sin(2\pi f_0 t)$	$\frac{1}{2j}\delta(f-f_0)-\frac{1}{2j}\delta(f+f_0)$
$e^{-a\lvert t\rvert},\quad a>0$	$\frac{2a}{a^2+(2\pi f)^2}$
$x(t)=\begin{cases}1-\lvert t\rvert & \lvert t\rvert\le 1\\ 0 & \lvert t\rvert>1\end{cases}$	$X(f)=\text{sinc}^2(f)$

<u>Parseval's Relation:</u> If $X(f)$ is the Fourier Transform of $x(t)$,

$$\int_{-\infty}^{\infty}\lvert x(t)\rvert^2 dt=\int_{-\infty}^{\infty}\lvert X(f)\rvert^2 df$$

<u>Useful Facts:</u>

1. A random variable is jointly Gaussian with itself.
2. If X_1, X_2, X_3, X_4 are jointly Gaussian random variables, each with zero mean, then:

$$E[X_1X_2X_3X_4]=E[X_1X_2]E[X_3X_4]+E[X_1X_3]E[X_2X_4]+E[X_1X_4][X_2X_3]$$

1. [20] Consider the region $[0,1] \times [0,1]$ in the (x,y)-plane; that is, the square with the point (0,0) at its lower left corner left and the point (1,1) as its upper right corner. Suppose I pick a point at random in this square (all points equally likely) and let the outcome of the experiment be the *Manhattan distance* $|x|+|y|$ of that point from (0,0). Find a probability space $(S, \mathcal{A}, P)$ for this experiment. (**Note: Full credit goes to the most useful space, so do not use a trivial $\mathcal{A}$, such as $\mathcal{A} = \{0, S\}$.**)

2. Suppose that I conduct the following experiment. I flip a coin continually. Whenever I get a "Tail", I write down a "0". Whenever I get a "Head", I write down a "1". The resulting sequence of 0s and 1s forms the outcome ω. Hence, a possible ω might look like:

$$\omega = (0, 0, 0, 1, 1, \ldots)$$

if the flips were "Tail", "Tail", "Tail", "Head", "Head", ...

[10] (a) Consider the sample space S of all possible outcomes ω. Is S countable or uncountable?

[10] (b) Index the components of ω in the obvious way; that is, $\omega = (\omega_1, \omega_2, \ldots)$, where $\omega_1 \in \{0,1\}$ is from the first flip, $\omega_2 \in \{0,1\}$ is from the second flip, etc. Let $Y_n(\omega) = (1.5)^n \prod_{i=1}^{n} \omega_i = (1.5)^n \, \omega_1 \cdot \omega_2 \cdot \omega_3 \cdot \ \ldots \ \cdot \omega_n$. Does Y_n converge? If so, to what, and in which ways?

[10] (c) Now let $Z_n(\omega)$ be defined as follows:

$$Z_n(\omega) = \begin{cases} 1, & n \text{ odd}, \ \omega_1 = 0 \\ 1, & n \text{ even}, \ \omega_1 = 1 \\ 0, & \text{otherwise} \end{cases}$$

In other words, the sequence Z_n only depends on the outcome ω_1 of the first flip. If the first flip is "Tails", $(Z_n) = (1, 0, 1, 0, \ldots)$. If the first flip is "Heads", $Z_n = (0, 1, 0, 1, \ldots)$. Does Z_n converge? If so, to what and in which ways?

3. The random variables X and Y have joint probability density function give by:

$$f_{X,Y}(x,y) = \begin{cases} c, & -1 \le x \le 1, \ 0 \le y \le 1 - |x| \\ 0, & \text{else} \end{cases}$$

[5] (a) Find c.

[10] (b) Find $f_{X|Y}(x|y)$, the conditional density function of X given Y.

[5] (c) Find $P(\{X > \frac{1}{4}\} | \{Y = \frac{1}{2}\})$.

4. Let $Z(t)$ be a Gaussian random process with zero-mean and autocorrelation function $R_Z(\tau) = \frac{1}{2}e^{-\frac{|\tau|}{4}}$.

[10] (a) Suppose I take a sample at time $t = 0$; call it $X_1 = Z(0)$. I run this sample through a square-law device to yield $Y = X_1^2 + 2$. Find $P(Y > 3)$.

[10] (b) Suppose that I form the linear combination $X_2 = \frac{1}{2}Z(0) + \frac{1}{2}Z(1)$. Find $f_{X_2}(x)$, the probability density function (pdf) of X_2, and calculate $P(X_2 > 1)$.

5. Consider zero-mean stationary Gaussian random processes $X(t)$ and $Y(t)$, with respective power spectral densities $S_X(f)$ and $S_Y(f)$. Suppose that $X(t)$ and $Y(t)$ are independent.

[10] (a) Let $Z(t) = X(t) + Y(t)$. Find the power spectral density of $Z(t)$ in terms of $S_X(f)$ and $S_Y(f)$. **Show your work.**

[10] (b) Let $Z(t) = X(t) \cdot Y(t)$. Find the power spectral density of $Z(t)$ in terms of $S_X(f)$ and $S_Y(f)$. **Show your work.**