ECE 603 - Probability and Random Processes, Fall 2008

Final Exam

December 15th, 10:30am-12:30, LGRT 0321

Overview

- The exam consists of five problems for 110 points. The points for each part of each problem are given in brackets - you should spend your **two hours** accordingly.
- The exam is closed book, but you are allowed **three page-sides** of notes. Calculators are not allowed. I will provide all necessary blank paper.

Testmanship

- **Full credit will be given only to fully justified answers.**
- Giving the steps along the way to the answer will not only earn full credit but also maximize the partial credit should you stumble or get stuck. If you get stuck, attempt to neatly define your approach to the problem and why you are stuck.
- If part of a problem depends on a previous part that you are unable to solve, explain the method for doing the current part, and, if possible, give the answer in terms of the quantities of the previous part that you are unable to obtain.
- Start each problem on a new page. Not only will this facilitate grading but also make it easier for you to jump back and forth between problems.
- If you get to the end of the problem and realize that your answer must be wrong, be sure to write "this must be wrong because . . . " so that I will know you recognized such a fact.
- Academic dishonesty will be dealt with harshly - the *minimum penalty* will be an "F" for the course.

Some potentially useful information

$$\cos(\theta) = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$$
$$\sin(\theta) = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$$

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$$
$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$$

$$\cos(a)\cos(b) = \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right]$$
$$\sin(a)\sin(b) = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$$
$$\sin(a)\cos(b) = \frac{1}{2}\left[\sin(a-b) + \sin(a+b)\right]$$

Time Function	**Fourier Transform**
$x(at+b)$	$\frac{1}{\lvert a\rvert} X\left(\frac{f}{a}\right) e^{j2\pi \frac{b}{a} f}$
$p(t) = \begin{cases} 1 & \lvert t \rvert \le 1/2 \\ 0 & \lvert t \rvert > 1/2 \end{cases}$	$P(f) = \frac{\sin(\pi f)}{\pi f} = \text{sinc}(f)$
$\cos(2\pi f_0 t)$	$\frac{1}{2}\left[\delta(f-f_0) + \delta(f+f_0)\right]$
$\sin(2\pi f_0 t)$	$\frac{1}{2j}\delta(f-f_0) - \frac{1}{2j}\delta(f+f_0)$
$e^{-a\lvert t\rvert}, \quad a > 0$	$\frac{2a}{a^2+(2\pi f)^2}$
$x(t) = \begin{cases} 1-\lvert t\rvert & \lvert t\rvert \le 1 \\ 0 & \lvert t\rvert > 1 \end{cases}$	$X(f) = \text{sinc}^2(f)$

Parseval's Relation: If $X(f)$ is the Fourier Transform of $x(t)$,

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

Useful Facts:

1. A random variable is jointly Gaussian with itself.
2. If X_1, X_2, X_3, X_4 are jointly Gaussian random variables, each with zero mean, then:

$$E[X_1X_2X_3X_4] = E[X_1X_2]E[X_3X_4] + E[X_1X_3]E[X_2X_4] + E[X_1X_4][X_2X_3]$$

1. For the probability space $(\Omega, \mathcal{F}, P)$, let $\Omega = (0,1)$ and $\mathcal{F} = \mathcal{B}$ (restricted to $(0,1)$, of course). The subset $S(a,b,c,d)$ of $(0,1)$ is defined as the interval (a,d), with the removal of the set of rational numbers in (a,b), and with the removal of the set of irrational numbers in (c,d), where $0 \leq a < b < c < d \leq 1$.

   [10] (a) Starting from the basic definition that the Borel field is the σ-algebra generated from all intervals (a,b) such that $0 < a < b < 1$, show that $S(a,b,c,d)$ is an element of the Borel field.

   [5] (b) Now, define $P((a,b)) = b - a$. Find $P(\omega \in S(a,b,c,d))$.

   [5] (c) Now, define

$$P((a,b)) = \begin{cases} \frac{b^2-a^2}{2}, & 0 \leq a < b \leq \frac{3}{4} \\ \frac{b^2-a^2}{2} & \frac{3}{4} \leq a < b \leq 1 \\ \frac{1}{2} + \frac{b^2-a^2}{2}, & a < \frac{3}{4} < b \leq 1 \end{cases}$$

   Find $P(\omega \in S(0, 1/4, 1/2, 1))$.

2. *A single random variable:*

   [5] (a) Let X be uniformly distributed on the interval $[-1, 2]$. Find the probability density function $f_Y(y)$ of the random variable $Y = -X$. **A very short sentence of justification is sufficient.**

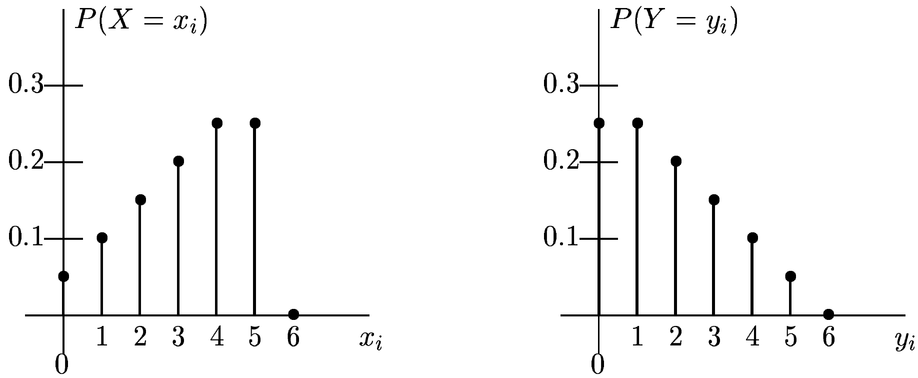
   [7] (b) Let the density of the random variable X be given by:

$$f_X(x) = \begin{cases} 2(1-x), & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

   For a random variable X, the **median** of X is defined to be the value m such that $P(X \leq m) = \frac{1}{2}$. Find the median of X.

   [8] (c) For an arbitrary random variable X with probability density function $f_X(x)$, the minimum mean squared estimate (MMSE) estimate of X (without any other information such as Y) is the value x_0 the minimizes $E[(X - x_0)^2]$. Show that $x_0 = E[X]$.

3. Let X be a random variable representing the number of bad chips in a shipment shipped from Xcompany, and let Y be a random variable representing the number of bad chips in a shipment shipped from Ycompany. The probability assignment functions of these two random variables are given below:



The probability that a shipment (or set of shipments) comes from from Xcompany is 0.25 and the probability a shipment (or set of shipments) comes from Ycompany is 0.75. Given the company, different shipments contain an independent number of bad chips. *Note: Some of the fractions get a little messy - sorry!*

[6] (a) Assume you receive a single shipment. Find the probability that the number of bad chips in the shipment is less than or equal to 3.

[6] (b) Given the number of bad chips in the shipment is less than or equal to 3, find the probability that it came from Ycompany.

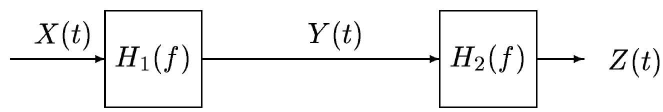
[6] (c) Suppose that you are going to receive a set of two different shipments from the same company (Xcompany with probability 0.25 and Ycompany with probability 0.75). In the first shipment, you observe less than or equal to 3 bad chips in the shipment. What is the probability of less than or equal to 3 bad chips in the second shipment?

[7] (d) Suppose that you are going to receive ten shipments from the same company (Xcompany with probability 0.25 and Ycompany with probability 0.75). What is the probability that you observe less than or equal to 3 bad chips in at least half of the shipments? (**Just write the expression. No need to evaluate.**)

4. Let $X(t)$ be a wide-sense stationary Gaussian random process with mean $m_X(t) = 0$ and autocorrelation function $R_X(\tau) = 10\delta(\tau)$, where $\delta(\cdot)$ is the Dirac delta function. The random process $X(t)$ is run through a filter combination as shown below, where the first filter has frequency response:

$$H_1(f) = \begin{cases} 1, & |f| < 3 \\ 0, & \text{otherwise} \end{cases}$$

and the second filter has frequency response $H_2(f) = e^{-f^2/4}$.



[5] (a) Find the power $E[Y^2(t)]$ in $Y(t)$.

[10] (b) Find the power $E[Z^2(t)]$ in $Z(t)$. (*Note: that the integral should look familiar, so simplify your answer as much as possible.*)

5. Let $X(t)$ be a zero-mean wide-sense stationary Gaussian random process with autocorrelation function $R_X(\tau)$ and power spectral density $S_X(f)$.

[10] (a) Let $Y(t) = t^2 + X(t)$.

- Find the mean function $m_Y(t)$ and autocorrelation function $R_Y(t_1, t_2)$ for $Y(t)$ in terms of $R_X(\tau)$.
- Is $Y(t)$ wide-sense stationary? If so, find its power spectral density in terms of $S_X(f)$.
- Find the probability density function for $Y(2)$.

[20] (b) Let $Z(t) = X^2(t) - R_X(0)$.

- Find the mean function $m_Z(t)$ and autocorrelation function $R_Z(t_1, t_2)$ for $Z(t)$ in terms of $R_X(\tau)$.
- Is $Z(t)$ wide-sense stationary? If so, find its power spectral density in terms of $S_X(f)$.
- Find the probability density function for $Z(2)$. (*Hint: This is going to take you a little bit of work.*)