ECE 603 - Probability and Random Processes, Fall 2006

Final Exam

December 15th, 10:30am-12:30pm, Marston 132

Overview

- The exam consists of five problems for 120 points. The points for each part of each problem are given in brackets - you should spend your **two hours** accordingly.
- The exam is closed book, but you are allowed **three page-sides** of notes. Calculators are not allowed. I will provide all necessary blank paper.

Testmanship

- **Full credit will be given only to fully justified answers.**
- Giving the steps along the way to the answer will not only earn full credit but also maximize the partial credit should you stumble or get stuck. If you get stuck, attempt to neatly define your approach to the problem and why you are stuck.
- If part of a problem depends on a previous part that you are unable to solve, explain the method for doing the current part, and, if possible, give the answer in terms of the quantities of the previous part that you are unable to obtain.
- Start each problem on a new page. Not only will this facilitate grading but also make it easier for you to jump back and forth between problems.
- If you get to the end of the problem and realize that your answer must be wrong, be sure to write "this must be wrong because ..." so that I will know you recognized such a fact.
- Academic dishonesty will be dealt with harshly - the *minimum penalty* will be an "F" for the course.

Hint: You may find the following fact useful as you solve this exam:

Fact: We know from class that any linear combination $\sum_{i=1}^{N} \alpha_i Y_i$ of jointly Gaussian random variables $\{Y_1, Y_2, \ldots, Y_N\}$ is Gaussian. The converse is also true: If all linear combinations $\sum_{i=1}^{N} \alpha_i Y_i$ are Gaussian, then the random variables $\{Y_1, Y_2, \ldots, Y_N\}$ are jointly Gaussian.

1. I am running a machine that produces parts. The machine has good days and bad days. On a good day, each part produced has a 10% chance of being defective; on a bad day, each part produced has a 50% chance of being defective. **Given the type of day**, the event of a part being defective is independent of the event of any other part being defective on that day. The machine has good days 75% of the time and bad days 25% of the time.

   Give **expressions** for the following quantities:

   [5] (a) Assuming that today is a good day, find the probability that there are 5 defective parts in a batch of 20 parts produced by the machine today.

   [5] (b) Without knowledge of whether today will be a good day or a bad day, find the probability that there are 5 defective parts in a batch of 20 parts produced by the machine today.

   [7] (c) Suppose I test the first 5 parts made by the machine on a given day and find that 3 of the parts are defective. Find the probability that this is a good day and the probability that this is a bad day.

   [8] (d) Suppose I test the first 5 parts made by the machine on a given day and find that 3 of the parts are defective. Find the probability that there will be 5 defective parts in the first 20 parts made that day (including the 5 parts that you have tested).

2. The random variables X and Y have joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} cx^2y, & -1 \leq x \leq 1, 0 \leq y \leq |x| \\ 0, & \text{otherwise} \end{cases}$$

   [5] (a) Find the value of c.

   [10] (b) Find $f_{X|Y}(x|y)$, the conditional probability density function of X given $Y = y$. Sketch $f_{X|Y}(x|0.5)$ (i.e. the pdf for X given that $Y = 0.5$).

   [5] (c) Write an expression (no need to evaluate) for $P(Y < 1 - X)$.

   [10] (d) Define the function $g(\cdot)$ as:

$$g(x) = \begin{cases} x^2, & x < 0 \\ 0, & \text{else} \end{cases}$$

   Let $Z = g(X)$. Find $f_Z(z)$, the probability density function of Z.

3. You are working on a server and are studying the interarrival time of packets, which you decide to model as a random variable X. You know from experience that the probability density function of such interarrival times follows an exponential distribution:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases} .$$

Let the i^{th} interarrival time be given by X_i (assumed to have probability density function $f_X(x)$), and let all interarrival times be **independent**. You measure the first 1000 interarrival times and find that

$$\frac{1}{1000} \sum_{i=1}^{1000} X_i = 3.0$$

[8] (a) Estimate λ. **(Note: "Estimate" does <u>not</u> mean "guess"! Justify your answer.)**

[12] (b) Consider the sum $Z = \sum_{i=1001}^{2000} X_i$. Using your answer from (a), estimate the probability that $Z > 2500$?

[5] (c) Let $Y = X_{2001} - X_{2002}$. Using your answer from (a) (and perhaps part of that of (b)), find the variance of Y.

4. [15] The laws of large numbers tell us that for an independent and identically distributed sequence $\{X_i\}$, each with probability density function $f_X(x)$ with expected value $\mu = E[X]$ and $E[X^2] < \infty$, the sum

$$\frac{1}{N} \sum_{i=1}^{N} X_i$$

converges to its mean μ. The strong law of large numbers tells us that it converges almost surely. How else does it converge? (i.e. in probability, in distribution, in mean square). Be sure to justify your result.

5. Let T be a Gaussian random variable with mean 1 and variance 4. I observe the random process $X(t) = t - T$.

[5] (a) Find the probability density function $f_{X(t)}(x)$ for all t.

[10] (b) Find the mean function and autocorrelation function for $X(t)$.

[10] (c) Determine whether $X(t)$ is a Gaussian random process.