ECE 603 - Probability and Random Processes, Fall 2002

Final Exam

December 17th, 10:30am-12:30pm, Paige 202

Overview

- The exam consists of six problems for 120 points. The points for each part of each problem are given in brackets - you should spend your **two hours** accordingly.
- The exam is closed book, but you are allowed **three page-sides** of notes. Calculators are not allowed. I will provide all necessary blank paper.

Testmanship

- **Full credit will be given only to fully justified answers.**
- Giving the steps along the way to the answer will not only earn full credit but also maximize the partial credit should you stumble or get stuck. If you get stuck, attempt to neatly define your approach to the problem and why you are stuck.
- If part of a problem depends on a previous part that you are unable to solve, explain the method for doing the current part, and, if possible, give the answer in terms of the quantities of the previous part that you are unable to obtain.
- Start each problem on a new page. Not only will this facilitate grading but also make it easier for you to jump back and forth between problems.
- If you get to the end of the problem and realize that your answer must be wrong, be sure to write “this must be wrong because . . . ” so that I will know you recognized such a fact.
- Academic dishonesty will be dealt with harshly - the *minimum penalty* will be an “F” for the course.

Some potentially useful information

$$\cos(\theta) = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right)$$
$$\sin(\theta) = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$$

$$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$$
$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$$

$$\cos(a)\cos(b) = \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right]$$
$$\sin(a)\sin(b) = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$$
$$\sin(a)\cos(b) = \frac{1}{2}\left[\sin(a-b) + \sin(a+b)\right]$$

Time Function	**Fourier Transform**
$x(at+b)$	$\frac{1}{\lvert a\rvert}X\left(\frac{f}{a}\right)e^{j2\pi\frac{b}{a}f}$
$p(t) = \begin{cases} 1 & \lvert t\rvert \leq 1/2 \\ 0 & \lvert t\rvert > 1/2 \end{cases}$	$P(f) = \frac{\sin(\pi f)}{\pi f} = \text{sinc}(f)$
$\cos(2\pi f_0 t)$	$\frac{1}{2}\left[\delta(f-f_0) + \delta(f+f_0)\right]$
$\sin(2\pi f_0 t)$	$\frac{1}{2j}\delta(f-f_0) - \frac{1}{2j}\delta(f+f_0)$
$e^{-a\lvert t\rvert}, \quad a > 0$	$\frac{2a}{a^2+(2\pi f)^2}$
$x(t) = \begin{cases} 1-\lvert t\rvert & \lvert t\rvert \leq 1 \\ 0 & \lvert t\rvert > 1 \end{cases}$	$X(f) = \text{sinc}^2(f)$

Parseval's Relation: If $X(f)$ is the Fourier Transform of $x(t)$,

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

1. [15] Two points are drawn independently and at random from the interval $[0, 2]$. The outcome (or observation) for the experiment is the distance between the two points. Define a non-trivial probability space for this experiment; that is, find $(\Omega, \mathcal{A}, P)$, where Ω is the observation space, $\mathcal{A}$ is a set of subsets of Ω to which probabilities are assigned, and P is a probability mapping from $\mathcal{A}$ to $[0, 1]$.

2. Random variables X and Y have joint probability density function:

$$f_{X,Y}(x, y) = \begin{cases} c\,y, & -1 \le x \le 1, 0 \le y \le x^2, \\ 0, & \text{otherwise} \end{cases}$$

   where c is a constant.

   [5] (a) Find c.

   [5] (b) Find $f_X(x)$, the marginal probability density function of X.

   [5] (d) Find $P(X > Y)$, the probability that X is greater than Y.

   [10] (c) Find $f_{X|Y}(x|y)$, the conditional probability density function for X given $Y = y$ (be sure to give limits!).

3. You are measuring a random variable X in a field experiment and find that $E[X] = 2$ and $E[X^2] = 8$.

   [8] (a) Suppose that the random variable X is input to your system, which will malfunction if $X \ge 12$. What can you say about $P(X \ge 12)$, the probability that X is greater than or equal to 12?

   [7] (b) Repeat part (a), but now assume you have one additional piece of information: you know that X is Gaussian.

4. [15] The random process $X(n)$ is generated as follows: I flip a fair coin repeatedly. If the first n flips are "heads", I let $X(n) = 2^n$; however, if any one of the flips before time n is a "tail", $X(n) = 0$. (Another way to describe the same experiment: I flip a fair coin repeatedly as long as I get "heads" and record $X(n) = 2^n$ for time n; however, as soon as I get the first "tail", I then let $X(n) = 0$ for all n after that time). Does this sequence of random variables converge? If so, in what ways and to what limiting random variable do they converge?

5. Recall that the pulse function $p(t)$ is defined by:

$$p(t) = \begin{cases} 1, & t \leq \frac{1}{2} \\ 0, & \text{else} \end{cases}$$

[8] (a) A friend tells you that he/she has a wide-sense stationary random process $X(t)$ that is perfectly correlated over short intervals but then decorrelates; in fact, he/she claims that $R_X(\tau) = p(\tau)$. Can such a process exist? (Either give an example of such a process or show that such a process cannot exist).

[15] (b) Let $X(t)$ be a zero-mean wide-sense stationary random process with autocorrelation function $R_X(\tau) = e^{-3|\tau|}$, and let $Y(t) = \int_0^t X(s)ds$.

- Find the autocorrelation function $R_Y(t_1, t_2) = E[Y(t_1)Y(t_2)]$. Note that this gets complicated, but carry it as far as possible. Some of you will be able to carry it the whole way; if not, indicate how you would proceed from where you stopped.
- Is the process $Y(t)$ wide-sense stationary? Be sure to fully justify your answer.

[12] (b) Let $X(t)$ be a zero-mean wide-sense stationary random process with autocorrelation function $R_X(\tau) = e^{-3|\tau|}$, and let $Y(t) = \frac{1}{100} \int_{t-100}^t X(s)ds$.

- Is $Y(t)$ wide-sense stationary? If so, find its power spectral density $S_Y(f)$. If not, find the autocorrelation function $R_Y(t_1, t_2) = E[Y(t_1)Y(t_2)]$.
- Estimate the power in $Y(t)$. (Note: Estimate does *not* mean guess. There are multiple correct answers, but you must fully justify your answer.)

6. **Note that you do not have to know anything about estimation to solve this problem!**

[15] The maximum-likelihood estimator (slightly modified to fit this problem) is given as follows:

$$\hat{X}_{ML}(Y = y) = \text{argmax}_x P(Y = y|X = x)$$

We study a Poisson (point) process for which the arrival rate λ is unknown. Let Y be the number of arrivals during 5 seconds of observing the process. Find the maximum-likelihood (ML) estimate of λ given that $Y = 10$. [Be sure to fully justify your answer].