Midterm Solutions

1) (a) $$H(X) = -\sum_{x_k \in X} p_X(x_k)\log_2 p_X(x_k)$$

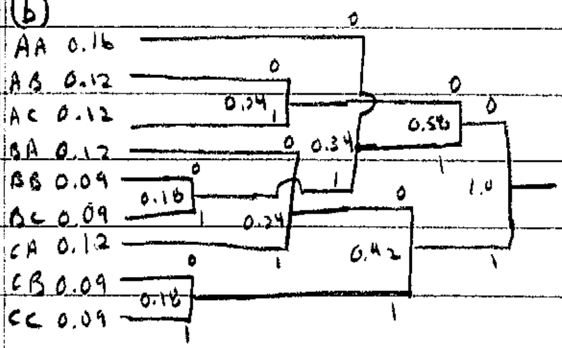
$$= -0.4\log_2 0.4 - 0.3\log_2 0.3 - 0.3\log_2 0.3$$

$$= 1.57 \text{ bits/symbol}$$

$$H(X) \le R_{opt} \le H(X) + 1/N$$

$$\Rightarrow 1.57 \le R_{opt} \le 1.57 + 1/N$$

(b)



AA	0.16	010
AB	0.12	000
AC	0.12	001
BA	0.12	100
BB	0.09	0110
BC	0.09	0111
CA	0.12	101
CB	0.09	110
CC	0.09	111

$$R = \frac{3\times 0.82 + 4\times 0.18}{2}$$

$$= 1.59 \text{ bits/symbol}$$

$$1.57 \le 1.59 \le 2.07 \checkmark$$

(c) For $N=2$

$$\frac{H_1(X_1) + H(X_2|X_1)}{2} \le R \le \frac{H(X_1) + H(X_2|X_1) + 1}{2}$$

$$1.57 + H(X_2|X_1) \le 1.8 \le 1.57 + H(X_2|X_1) + 1$$

$$\Rightarrow -0.77 \le H(X_2|X_1) \le 0.23$$

$$\Rightarrow 0 \le H(X_2|X_1) \le 0.23$$

and we know:

$$H(X_1, X_2, \ldots, X_N) \le N\cdot R_{opt} \le H(X_1, X_2, \ldots, X_N) + 1$$

$$\Rightarrow 0 \le N\cdot R_{opt} \le H(X_1) + H(X_2|X_1) + H(X_3|X_1,X_2) + \ldots + H(X_N|X_1,X_2,\ldots,X_{N-1}) + 1$$

$$\le H(X_1) + (N-1)H(X_2|X_1) + 1$$

$$\le 1.57 + 0.23(N-1) + 1$$

$$\Rightarrow 0 \le R \le 1.57/N + 0.23(N-1)/N + 1/N$$

2) (a) First, select $\varepsilon = 0.01$

Now, choose N large enough such that

$$P\left(A_{0.01}^{(N)}\right) \geq 1 - 10^{-10}$$

This gives us our typical set. Then encode/decode as in class: encode only the typical set and map everybody else to a single sequence

(b) $$H(X) = -\sum_{x_k \in \mathcal{X}} p_X(x_k) \log_2 p_X(x_k) = 2.023 \text{ bits/symbol}$$

Need to find the size of $B_\varepsilon^{(N)}$

Use ideas from class:

$$1 \geq \sum_{\underline{x} \in B_\varepsilon^{(N)}} p_{\underline{X}}(\underline{x})$$

$$\geq \sum_{\underline{x} \in B_\varepsilon^{(N)}} 2^{-N(H(X)+\varepsilon)} / c$$

$$= |B_\varepsilon^{(N)}| \, 2^{-N(H(X)+\varepsilon)} / c$$

$$\Rightarrow |B_\varepsilon^{(N)}| \leq c \, 2^{+N(H(X)+\varepsilon)}$$

$$\Rightarrow R \leq \frac{1}{N} \log_2 |B_\varepsilon^{(N)}| \leq \frac{1}{N}\left(\log_2 c + N(H(X)+\varepsilon)\right)$$

$$= H(X) + \varepsilon + \frac{\log_2 c}{N}$$

$\Rightarrow$ for large N, no effect on rate

(c) codeword length: $N(H(X)+\varepsilon) + \log_2\left(N(H(X)+\varepsilon)\right)$

$$\Rightarrow R = H(X) + \varepsilon + \frac{\log_2\left(N(H(X)+\varepsilon)\right)}{N}$$

Since the last term $\to 0$ as $N \to \infty$, no impact on achievable rate, but we will need to choose a (slightly) smaller ε - say $\varepsilon/2$.

3) (a)

$$H(X) = -\sum_{x_k \in \mathcal{X}} p_X(x_k) \log_2 p_X(x_k) = \log_2 3 = 1.58 \text{ bits/symbol}$$

$$\Rightarrow 15{,}800 \text{ bits/s}$$

$$C_{BSC} = 1 - H_2(p)$$

$$= 1 + 0.9 \log_2 0.9 + 0.1 \log_2 0.1 = 0.531 \text{ bits/symbol}$$

$$\Rightarrow 21{,}240 \text{ bits/s}$$

Yes! Choose R somewhere in between and then follow constructions from class.

(b)

No. For <u>any</u> channel output sequence, there is some (small) probability it came from any input; hence $P_E > 0$ for <u>any</u> rate.

(c) $I(X;Y) = H(Y) - \cancel{H(Y|X)}^{\,1}$

$$\leq \log_2 4 - 1 = 1$$

and I can achieve by using:

$$p_X(0) = 1/2 \quad p_X(1) = 0 \quad p_X(2) = 1/2 \quad p_X(3) = 0$$

and, note that if I use these input distributions, I can get 1 bit/channel use with

<u>no</u> <u>errors</u>.

4) (a)

$$h(Z_1, Z_2) = h(Z_1) + h(Z_2 | Z_1)$$

$$= \tfrac{1}{2}\log_2(2\pi e) + \tfrac{1}{2}\log_2(2\pi e(1-\rho^2))$$

$$= \tfrac{1}{2}\log_2(4\pi^2 e^2(1-\rho^2))$$

(b)

$$I(X;(Y_1,Y_2)) = h(\underline{Y}) - h(\underline{Y}|X)$$

$$= h(\underline{Y}) - h(\underline{Z})$$

$$= h(\underline{Y}) - \tfrac{1}{2}\log_2(4\pi e^2(1-\rho^2))$$

Now let's calculate statistics of $\underline{Y}$

$$Y_1 = X + Z_1$$
$$Y_2 = X + Z_2$$

$$E[Y_1^2] = E[(X+Z_1)^2] = E[X^2] + \cancelto{0}{2E[XZ_1]} + E[Z_1^2] \le P + 1$$

$$E[Y_2^2] = E[(X+Z_2)^2] \le P+1$$

$$E[Y_1Y_2] = E[(X+Z_1)(X+Z_2)] = E[X^2] + E[XZ_1] + E[XZ_2] + E[Z_1Z_2] = P + \rho$$

$$\Rightarrow K_Y = \begin{bmatrix} P+1 & P+\rho \\ P+\rho & P+1 \end{bmatrix}$$

This is fixed for any distribution. Thus, $\underline{Y}$ Gaussian maximizes $h(\underline{Y})$; hence, choose $X \sim N(0,P)$ and

$$C = \tfrac{1}{2}\log_2 4\pi e^2 \begin{vmatrix} P+1 & P+\rho \\ P+\rho & P+1 \end{vmatrix} - \tfrac{1}{2}\log_2(4\pi e^2(1-\rho^2))$$

$$= \tfrac{1}{2}\log_2\left(\frac{2P+1-2P\rho-\rho^2}{1-\rho^2}\right) = \tfrac{1}{2}\log_2\left(1 + \frac{2P}{1+\rho}\right)$$

(c) standard Gaussian $C = \tfrac{1}{2}\log_2(1+P)$

5)

<u>Idea</u>: Encode <u>a</u> <u>lot</u> of information bits in each symbol and repeat it until you are sure it got across.

<u>Solution</u>:

Need $P_E < \delta$. So choose N large enough such that probability that two go through the "good" channel is high enough.

$$2^{-N/2} < \delta/2$$

is sufficient. One symbol goes through "good" channel in first $N/2$ times with probability $> 1-\delta/2$, and same in second $N/2$. <u>Decoder</u>: look over block of length N for Y_i's that match. With probability 1, only match comes from those symbols that went through "good" channel.

<u>Encoder</u>: Need R bits/symbol or NR bits per codeword. Thus, choose X_i between 0 and $\sqrt{P}$ (guaranteeing $E[X^2] \le P$) as:

$$X = 0.\underbrace{00\ldots0}_{\substack{\text{enough to} \\ \text{always make } X_i \le \sqrt{P}}} b_1 b_2 b_3 \ldots b_{NR}$$

and repeat the symbol N times on the channel:

$$\underbrace{X, X, X, \ldots, X}_{N}$$