ECE 745 - Advanced Communication Theory, Spring 2009

Midterm Exam

Wednesday, April 8th, 6:30-9:30pm, Marcus 215

Overview

- The exam consists of five problems for 135 points. The points for each part of each problem are given in brackets **-** you should spend your **three hours** accordingly.
- The exam is closed book, but you are allowed **three page-side** of notes. Calculators are allowed for simple calculations (finding entropies, etc). I will provide all necessary blank paper.

Testmanship

- **Full credit will be given only to fully justified answers.**
- Giving the steps along the way to the answer will not only earn full credit but also maximize the partial credit should you stumble or get stuck. If you get stuck, attempt to neatly define your approach to the problem and why you are stuck.
- If part of a problem depends on a previous part that you are unable to solve, explain the method for doing the current part, and, if possible, give the answer in terms of the quantities of the previous part that you are unable to obtain.
- Start each problem on a new page. Not only will this facilitate grading but also make it easier for you to jump back and forth between problems.
- If you get to the end of the problem and realize that your answer must be wrong (e.g. the Fourier Transform of a real signal that is not conjugate symmetric), be sure to write “this must be wrong because . . . ” so that I will know you recognized such a fact.
- Academic dishonesty **will** be dealt with **harshly -** the *minimum penalty* will be an “F” for the course.

1. Consider a stationary source that generates a sequence (X_i) of identically distributed random variables, with each $X_i \in \mathcal{X} = \{A, B, C\}$ with probability distribution $P(X_k = A) = 0.4, P(X_k = B) = 0.3, P(X_k = C) = 0.3$.

   [8] (a) Suppose that the source is *independent* and identically distributed (IID). What can you say about the rate of an optimal lossless (source) code for this source as a function of N?

   [12] (b) Suppose that the source is independent and identically distributed (IID).

   - Find a Huffman code that takes blocks of $N = 2$ source symbols at a time.
   - Calculate the rate of your code in bits/symbol, and verify that it agrees with your result from part (a).

   [10] (c) Suppose that the stationary sequence (X_i) is still identically distributed, with the distribution of each X_i as given above, but now you have no idea what the relation is between different symbols (independent, correlated, etc). However, somebody who does know that relation designs a Huffman code that takes blocks of $N = 2$ source symbols, and the rate of that code is 0.9 bits/symbol.

   - Find upper and lower bounds to $H(X_2|X_1)$.
   - What can you say about the rate of an optimal lossless (source) code for this source as a function of N?

2. A source outputs an independent and identically distributed (IID) sequence $\{X_i\}$ of random variables, each drawn from alphabet $\mathcal{X} = \{x_0, x_1, x_2, x_3, x_4, x_5\}$, according to the following distribution:

Value	$P(x_i)$
x_0	0.05
x_1	0.05
x_2	0.15
x_3	0.2
x_4	0.5
x_5	0.05

   [10] (a) We know that we can code this source almost losslessly by employing a large N fixed-length block code as long as the rate is greater than $H(X)$. Suppose your boss wants a probability of error of 10^{-10} and a rate $R = H(X) + 0.01$. Give the construction of how you would go about designing a source coder and decoder, being sure to specify (or at least tell how to get) key parameters such as ϵ and N, and the encoding and decoding operations.

   [10] (b) Your boss gets nervous about you only coding the typical set, and says that, for a given ϵ, he would like you to encode a larger set of sequences (that includes the typical set) defined by:

$$B_\epsilon^{(N)} = \{\underline{x} : 2^{-N(H(X)+\epsilon)}/c \quad \leq \quad p_{\underline{X}}(x) \quad \leq \quad c\, 2^{-N(H(X)-\epsilon)}\}$$

   where $c > 1$ is some constant. Fix ϵ to be the value from (a), and find the rate (in bits/symbol) of the resulting almost lossless fixed-length source code.

   [10] (c) Your boss drops the idea in (b), but now he has a different problem. For any source coder output codeword length M, the system requires that we insert $\log_2 M$ fixed synchronization symbols (say, all 1's) at the beginning of each codeword. Can he still achieve the rate of part (a)? If so, indicate how it changes the construction. If not, indicate the lowest rate that you can achieve.

3. Company WirelessX has mastered the ability to build circuits of any complexity, and thus they hire you because of your knowledge of Information Theory.

[10] (a) The company desires to send an independent and identically distributed (IID) source, which generates symbols $X_1, X_2, X_3, \ldots$ at 10,000 symbols per second, and that is defined by the probability distribution:

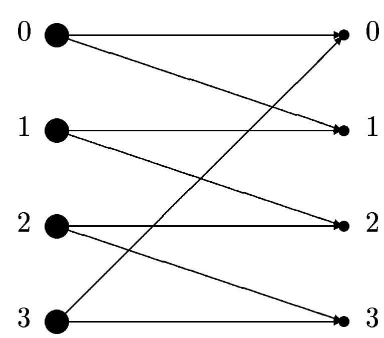
$$P(X_i = a) = \frac{1}{3}$$
$$P(X_i = b) = \frac{1}{3}$$
$$P(X_i = c) = \frac{1}{3}$$

You have access to a binary symmetric channel (BSC) with cross-over probability 0.9, which you are able to use 40,000 times per second. Can this source be sent over this channel with error probability less than 10^{-10}? If not, explain why. If so, explain how one would go about doing this. Describe both the initial system construction (how codebooks are built, etc.) and also how an actual source message is transmitted through the system. *Obviously, you will not be able to find codebook lengths, etc, but be as clear and precise as possible.*

[10] (b) Suppose now that the customer wants no errors; that is, the probability must be *exactly* zero in your system from the transmitter to your receiver.

- Can the source in part (a) be sent across the channel from part (a)?
- What is the maximum entropy rate of a source that can be send across the channel from part (a) under the "no error" criterion?

[10] (c) Now consider the discrete memoryless channel described by:



where each of the lines is labeled with a $\frac{1}{2}$.

- For a probability of error of 10^{-10}, what is the achievable rate (in bits/symbol) on such a channel? (i.e. in other words, what is the Shannon capacity?)
- Suppose the customer wants zero error probability (as in (b)). What is the achievable rate on this channel?

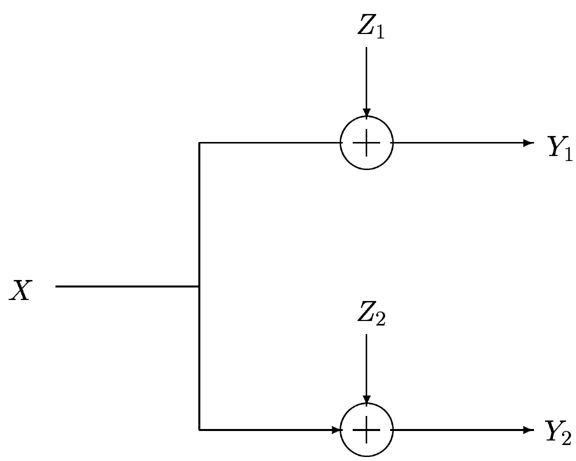
4. Let Z_1 and Z_2 be jointly Gaussian random variables, with $E[Z_1] = E[Z_2] = 0$, $E[Z_1^2] = E[Z_2^2] = 1$, and correlation coefficient $\rho = E[Z_1 Z_2]$ a parameter. You can assume $|\rho| < 1$ for parts (a) and (b).

   Fact: *Conditioned on Z_1, the random variable Z_2 is Gaussian with mean ρZ_1 and variance $1 - \rho^2$.*

   [8] (a) Find $h(Z_1, Z_2)$.

   [12] (b) Now consider the single-input, two-output channel shown below, where X is sent by the transmitter, and both Y_1 and Y_2 are observed by the receiver Assuming $|\rho| < 1$, find its information capacity under an input power constraint P:

$$C = \max_{f_X(x): E[X^2] \leq P} I(X; (Y_1, Y_2))$$



   [5] (c) Numerical problems make it difficult to follow the line of thinking of (a) and (b) when $\rho = 1$. But that is okay, because you should be able to write it down by inspection. What is the capacity of the channel from part (b) when $\rho = 1$?

5. [20] Consider a continuous-valued channel with input X_i and output $Y_i = X_i + Z_i$, where $\{Z_i\}$ is an independent and identically distributed (IID) sequence of strange "noise" variables, as follows. Any Z_i is generated as follows: I flip a coin. If the coin is "heads", Z_i = 0.0. If the coin is tails, $Z_i \sim N(0, \sigma^2)$. Suppose I have an input power constraint $E[X_i^2] \leq P$.

   For an arbitrarily small error probability $\delta > 0$, I claim that I can achieve any finite rate R bits/symbol on this channel - no matter how big! Give a construction for doing such; in other words, for arbitrary (large) finite R and arbitrary $\delta > 0$, give a method for achieving probability of error $< \delta$.