ECE 745 - Advanced Communication Theory, Spring 2009
Homework #2 [A couple problems to keep your mind active.]

Due: May 1, 2009

1. We know from class that the rate R_{HUFF} of a Huffman code that takes input blocks of $N = 1$ characters can be upper bounded as $R_{HUFF} \leq H(X) + 1$, where $H(X)$ is the first-order entropy of the source. Recall that this bound was obtained by considering Shannon-Fano codes. Note that such a bound does not require precise knowledge of the source distribution $\{p_1, p_2, \ldots, p_M\}$, where p_i is the probability of the i^{th} character, and M is the number of characters in the source alphabet. However, suppose we know the probability p_1 of the first character, and define a new lossless code with codeword lengths $\{l_1, l_2, \ldots, l_M\}$ as follows:

$$l_j = \begin{cases} \lceil -\log_2 \, p_1 \rceil, & j = 1 \\ \left\lceil -\log_2 \, \left(p_j \left(\frac{1 - 2^{-\lceil -\log_2 \, p_1 \rceil}}{1 - p_1} \right) \right) \right\rceil, & j > 1 \end{cases}$$

Note: Perhaps this is obvious, but the numerator of the innermost term of the second line is $1 - 2^{-\lceil -\log_2 \, p_1 \rceil}$ **(i.e. the** $-\lceil -\log_2 \, p_1 \rceil$ **is all in the exponent of** 2**).**

(a) Show that $1 \leq l_j \leq \lceil -\log_2 \, p_j \rceil$ for all j.

(b) Show that $\{l_1, l_2, \ldots, l_M\}$ satisfies the Kraft inequality.

(c) Obtain an upper bound on R_{HUFF} in terms of $H(X)$ and p_1 that is tighter than $H(X) + 1$.

(d) Making the obvious extension to $N = 2$, find this upper bound for an $N = 2$ Huffman code on the source of Problem 1 (note that you can pick any block to be your "p_1" block).

2. You are communicating on a channel with additive Gaussian noise:

$$Y_i = X_i + N_i$$

where X_i is the transmitted signal, Y_i is the received signal and N_i is the noise. The interesting thing about this channel is that it can be in one of two states: "Good" or "Bad" (e.g. a wireless channel). When the channel is in a "Good" state, $N_i \sim N(0, 1)$ and, when the channel is in a "Bad" state, $N_i \sim N(0, 6)$. In any case, the noise is independent across i.

The state of the channel at time i is given by a stationary random process S_i, where the state at time i depends only on the state at time $i - 1$. Its transitions are guided by:

$$\begin{aligned} P(S_i = \text{“Good”} | S_{i-1} = \text{“Bad”}) &= 0.45 \\ P(S_i = \text{“Bad”} | S_{i-1} = \text{“Bad”}) &= 0.55 \\ P(S_i = \text{“Good”} | S_{i-1} = \text{“Good”}) &= 0.7 \\ P(S_i = \text{“Bad”} | S_{i-1} = \text{“Good”}) &= 0.3 \end{aligned}$$

Both the transmitter and the receiver know the current channel state S_i, so the generation of a given symbol and its processing at the receiver can be based on the current state of the channel. For example (and this is intentionally a communications systems example rather than an information theory example, so do not let it confuse you), the transmitter could send a bit with +1 (bit=0) or -1 (bit=1) when the channel is "Good", and send a bit with +4 (bit=0) or -1 (bit=1) when the channel is "Bad". Because it knows the channel state, the receiver would know the corresponding mapping for any given symbol and be able to perform optimal decoding of each bit.

Now, turn your mind back to information theory (e.g. Gaussian codebooks, etc.). Given an average transmit power constraint $E[X_i^2] < 10$ (meaning that the power averaged over all i must be less than 10), provide a scheme that achieves as large an average rate as possible on this channel. Be sure to provide the details of all codebook constructions, powers employed, etc.