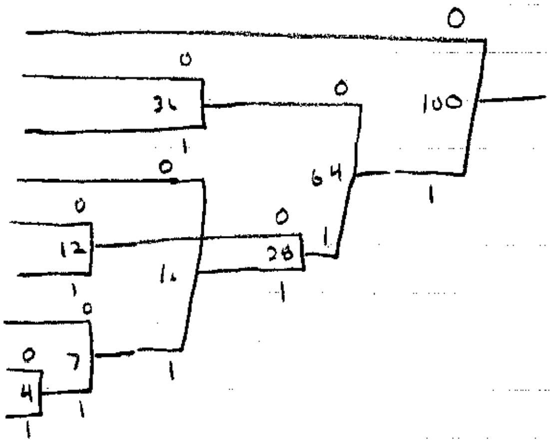
Final Exam Solutions

ECE 745 (These are my "sketch" solutions I use for grading. Yours should be more detailed). Spring, 2007

1) (a)

		$p_k * 100$
0	(A,A)	36
100	(A,B)	18
101	(B,A)	18
1110	(B,B)	9
1100	(A,C)	6
1101	(C,A)	6
11110	(B,C)	3
111110	(C,B)	3
111111	(C,C)	1



$$R = (36\cdot 1 + 2\cdot 18\cdot 3 + 4\cdot 9 + 2\cdot 6\cdot 4 + 5\cdot 3 + 6\cdot 3 + 6\cdot 1)/100/2$$

$$= 1.33 \text{ bits/input symbol}$$

$$H(x) = \sum_{x\in X} -p(x)\log_2 p(x) = 1.295 \ , \quad 1.295 \le R \le 1.795$$

(b) we know

$$H(x) \le R_{Huff} \le H(x) + 1/N$$

$$1.295 \le R_{Huff} \le 1.42$$

but the N=8 code cannot be worse than N=2; thus,

$$1.295 \le R_{Huff} \le 1.33$$

(if it were worse, using N=2 Huffman 4 times would yield a better code)

2) (a) First key is to note:

$$2^{-(-\log_2 p_1 + 1)} \le 2^{-\lceil -\log_2 p_1 \rceil} \le 2^{-(-\log_2 p_1)} = p_1$$

and thus (using the right)

$$\frac{1 - 2^{-\lceil -\log_2 p_1 \rceil}}{1 - p_1} \ge 1 \quad \Rightarrow \quad \ell_j \le \lceil -\log_2 p_j \rceil$$

and, using the left

$$p_j \left(\frac{1 - 2^{-\lceil -\log_2 p_1 \rceil}}{1 - p_1} \right) \le p_j \frac{(1 - p_1/2)}{(1 - p_1)} \le \left(\cancel{\sum_{j=2}^{m} p_j} \right) \left(\frac{1 - p_1/2}{\cancel{1 - p_1}} \right) = 1 - p_1/2$$

(b)
$$\sum_{j=1}^{m} 2^{-\ell_j} = 2^{-\lceil \log_2 p_1 \rceil} + \sum_{j=2}^{m} 2^{\lceil -\log_2 \left(p_j \left(\frac{1 - 2^{-\lceil -\log_2 p_1 \rceil}}{1 - p_1} \right) \right) \rceil}$$

$$\le 2^{-\lceil \log_2 p_1 \rceil} + \sum_{j=2}^{m} 2^{-\log_2 \left(p_j \left(\frac{1 - 2^{-\lceil -\log_2 p_1 \rceil}}{1 - p_1} \right) \right)}$$

$$= 2^{-\lceil \log_2 p_1 \rceil} + \underbrace{\left(\sum_{j=2}^{m} p_j \right)}_{1 - p_1} \left(\frac{1 - 2^{-\lceil -\log_2 p_1 \rceil}}{1 - p_1} \right)$$

$$= 1$$

(c) $R = E[\ell_j]$

$$= p_1 \lceil -\log_2 p_1 \rceil + \sum_{j=2}^{m} p_j \left\lceil -\log_2 \left(p_j \left(\frac{1 - 2^{-\lceil -\log_2 p_1 \rceil}}{1 - p_1} \right) \right) \right\rceil$$

$$\le p_1 (-\log_2 p_1 + 1) + \sum_{j=2}^{m} p_j \left[\left(-\log_2 \left(p_j \left(\frac{1 - 2^{-\lceil -\log_2 p_1 \rceil}}{1 - p_1} \right) \right) \right) + 1 \right]$$

$$= \sum_{j=1}^{m} p_j + \sum_{j=1}^{m} -p_j \log_2 p_j + \sum_{j=2}^{m} p_j \left(-\log_2 \left(\frac{1 - 2^{-\lceil -\log_2 p_1 \rceil}}{1 - p_1} \right) \right)$$

$$= 1 + H(x) - \underbrace{(1 - p_1)}_{\ge 0} \underbrace{\log_2 \left(\frac{1 - 2^{-\lceil -\log_2 p_1 \rceil}}{1 - p_1} \right)}_{\ge 0}$$

(d) applying to (1): pick any of the nine blocks

(e.g. $p_1 = 0.36$)

$$\leq 2H(x) + 1 - \log_2 \frac{\left(1 - 2^{-\lceil -\log_2 0.36 \rceil}\right)}{(1-0.36)} (1-p_1)$$

$$= 2H(x) + 1 - 0.23\,(0.64)$$

$$= 2H(x) + 0.8528$$

$$R \leq H(x) + 0.4264 = 1.7214$$

3) (b) Just like in our LZW class. Need $\lceil \log_2 k \rceil$ bits to send

$$\underbrace{00\ldots01}_{\lceil \log_2 k \rceil} \quad \underbrace{xxxx}_{\lceil \log_2 k \rceil \text{ (value)}}$$

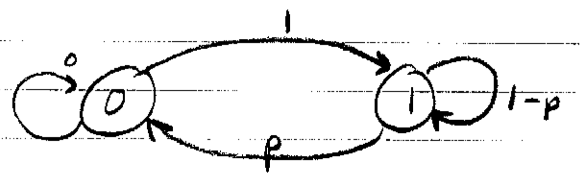
$$= 2\lceil \log_2 k \rceil + 1 \quad \text{bits}$$

But can encode $00\ldots0$ as a number that requires $2\lceil \log_2 \lceil \log_2 k \rceil \rceil + 1$

$$\leq \log_2 k + \log_2 \log_2 k + 4 \quad \text{bits}$$

(see example on next page)

(a) Need a source model. I assume this Markov model



where p is estimated from the data as:

$$p = \frac{\#0\text{'s after } 1}{\#1\text{'s}} = \frac{9}{90} = 0.1$$

Find the stationary distribution:

$$\begin{bmatrix} p_0 \\ p_1 \end{bmatrix} = \begin{bmatrix} 0 & 0.1 \\ 1 & 0.9 \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \end{bmatrix} \Rightarrow \begin{matrix} p_0 = 1/11 \\ p_1 = 10/11 \end{matrix} \quad \text{(as expected)}$$

$$H((X_i)) = H(X_2 | X_1) = E_{X_1}[H(X_2 | X_1 = x_1)]$$
$$= 0 \cdot 1/11 + H_2(0.9) \cdot 10/11 = 0.426$$

(c)

7 requires 3 bits 111

thus, first form

$\underbrace{000}_{3\ 0\text{'s}}1\ \ 111$

now, encode 3 (2 bits 11) as:

001 11

to yield

001 11 111

7	11	4	14	18
00111111	00011001011	00111100	00011001110	0001101110010

16	7	8	5
0001101110000	00111111	00011001000	00111101

(d) <u>Idea:</u>

Encoder turns run of k 1's into $\log_2 k$ 1's (roughly).

Thus, consider

$$\frac{E[\log_2 k]}{E[k+1]}$$

for k a geometric r.v. with parameter 0.9. This will not approach the entropy rate.

4)

(a)

First, write

$$I(X;Y) = H(Y) - H(Y|X)$$

$$= H(Y) - \mathcal{H}_2(p)$$

$$\leq 2 - \mathcal{H}_2(p)$$

By using equally likely inputs, I achieve $H(Y) = 2$, and thus

$$C = \max_{p_X(x)} I(X;Y) = 2 - \mathcal{H}_2(p)$$

(b)

$$2 - \mathcal{H}_2(p) = 1 + \underbrace{1 - \mathcal{H}_2(p)}_{\text{BSC}}$$

Signal at rate $1 - \mathcal{H}_2(p)$ bits/c.u. Encode one more bit/c.u. by using that bit to pick a BSC: either $\{0,1\} \to \{0,1\}$ or $\{2,3\} \to \{2,3\}$

At decoder, decode one bit/c.u. by just looking at whether received $\in \{0,1\}$ or received $\in \{2,3\}$. Then decode BSC output.

5)

First, find the stationary distribution:

$$\begin{bmatrix} p_G \\ p_B \end{bmatrix} = \begin{bmatrix} 0.7 & 0.45 \\ 0.3 & 0.55 \end{bmatrix} \begin{bmatrix} p_G \\ p_B \end{bmatrix} \Rightarrow \begin{matrix} p_G = 0.6 \\ p_B = 0.4 \end{matrix}$$

What I will do is use a Gaussian codebook of power P_G when the channel is good, P_B when the channel is bad.

$$R = 0.6 \cdot \tfrac{1}{2} \log_2\left(1 + P_G/1\right) + 0.4 \cdot \tfrac{1}{2} \log_2\left(1 + P_B/6\right)$$

but I know that $0.6 P_G + 0.4 P_B = 10$

$$\Rightarrow P_B = 25 - 1.5 P_G$$

$$\Rightarrow R = 0.3 \log_2(1 + P_G) + 0.2 \log_2\left(1 + 25/6 - P_G/4\right)$$

$$dR/dP_G = \frac{0.3}{1+P_G} - \frac{0.05}{31/6 - P_G/4} = 0$$

$$\frac{\cancel{0.3}\;1}{1+P_G} = \frac{\cancel{0.3}\;1}{31 - 3/2\,P_G}$$

$$1 + P_G = 31 - 3/2\,P_G$$

$$2.5 P_G = 30$$

$$\Rightarrow P_G = 12$$

$$P_B = 25 - 1.5 P_G = 7$$

$$R = 0.3 \log_2(1+12) + 0.2 \log_2(1 + 7/6) = 1.333$$

6) This is Slepian-Wolf followed by multiple-access (MA). Use Slepian-Wolf to figure out the region you <u>need</u>; then, use MA to figure out the power to get a point in that region.

$H(X) = 0.88$

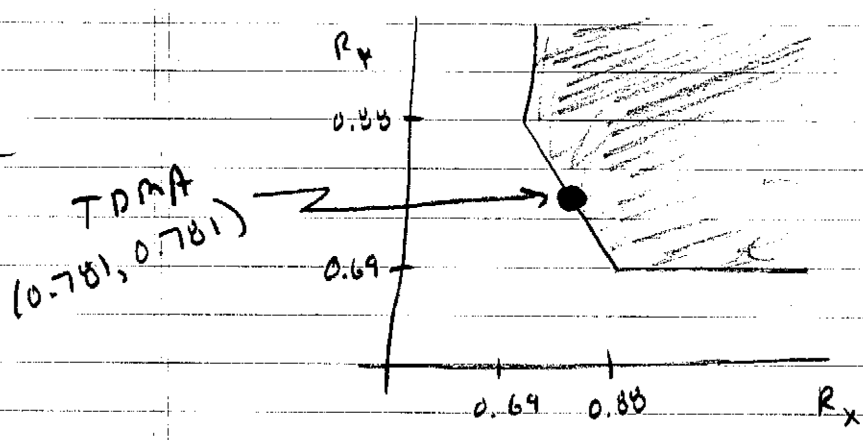
$H(X,Y) = 1.57$

$H(Y) = 0.88$

$H(X|Y) = H(Y|X) = H(X,Y) - H(X) = 0.69$

in this case

Thus, need a point in:



and want to find a MA region of minimum power that includes any point in above.

Mathematically:

$$R_X \le \tfrac{1}{2}\log_2(1 + P_X/N) \Rightarrow P_X \ge 2^{2(0.69)} - 1 = 1.602$$

$$R_Y \le \tfrac{1}{2}\log_2(1 + P_Y/N) \Rightarrow P_Y \ge 2^{2(0.69)} - 1 = 1.602$$

$$R_X + R_Y \le \tfrac{1}{2}\log_2(1 + (P_X + P_Y)/N) \Rightarrow P_X + P_Y \ge 2^{2(1.57)} - 1 = 7.81$$

$\Rightarrow P_X + P_Y = 7.81$. How to achieve it?

Note that TDMA at (0.785, 0.785) yields a constant power of $2^{2(1.57)} - 1 = 7.81$

test any point on the sum rate line works!

7) (a)

2 to each (cut in middle horizontally)

(b)

problem is only useful links from S_i to D_i are the middle two - only two useful packets per time slot!

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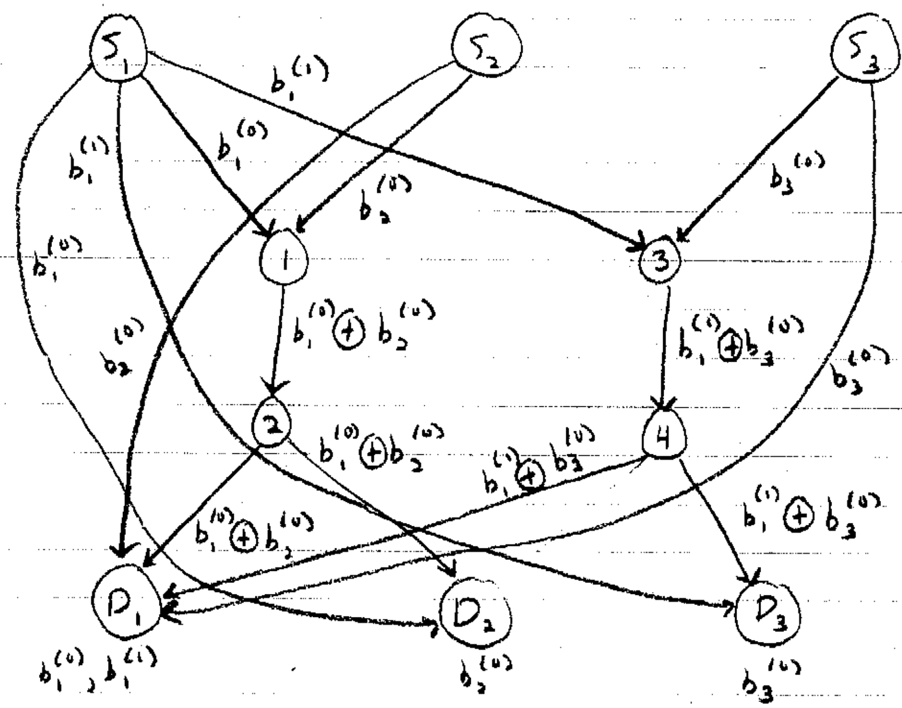
(c)



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(d) I did not grade this, except to give bonuses to good answers (some "yes" and some "no").