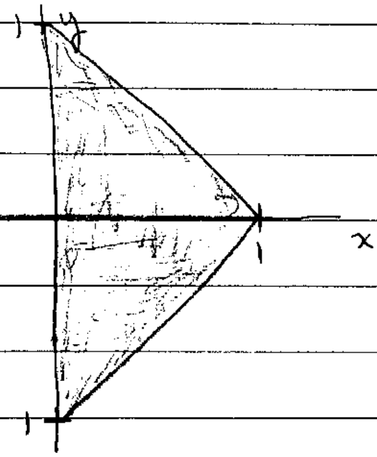
ECE 603 Fall, 2016

1)



(a) $1 = \int_0^1 \int_{x-1}^{1-x} c\,x\,dy\,dx$

$= \int_0^1 c\,x\,y \Big|_{x-1}^{1-x} dx$

$= \int_0^1 c\,x\,((1-x)-(x-1))\,dx$

$= 2c \int_0^1 (x - x^2)\,dx$

$= 2c\left(1/2 - 1/3\right) = c/3 \Rightarrow c = 3$

(b) $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy \overset{0 \le x \le 1}{=} \int_{x-1}^{1-x} 3x\,dy = 3xy \Big|_{x-1}^{1-x} = 6x - 6x^2$

$\Rightarrow f_X(x) = \begin{cases} 6x - 6x^2, & 0 \le x \le 1 \\ 0, & \text{else} \end{cases}$ Note: integrates to 1!

$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx \overset{0 \le y \le 1}{=} \int_0^{1-y} 3x\,dx = 3/2\, x^2 \Big|_0^{1-y} = 3/2\,(1-y)^2$

$\overset{-1 \le y \le 0}{=} \int_0^{1+y} 3x\,dx = 3/2\, x^2 \Big|_0^{1+y} = 3/2\,(1+y)^2$

$\Rightarrow f_Y(y) = \begin{cases} 3/2\,(1-|y|)^2, & -1 \le y \le 1 \\ 0, & \text{else} \end{cases}$

Check:
$\int_0^1 (3/2 - 3y + 3/2\,y^2)\,dy$
$= 3/2 - 3/2 + 1/2 = 1/2$ ✓

(c)

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\cancel{3}x}{\cancel{3}/2\,(1-|y|)^2} = \begin{cases} \dfrac{2x}{(1-|y|)^2}, & -1 \le y \le 1,\ 0 \le x \le 1-|y| \\ 0, & \text{else} \end{cases}$$

<u>Check:</u>

$$\int_0^{1-|y|} \frac{2x}{(1-|y|)^2}\,dx = \left.\frac{x^2}{(1-|y|)^2}\right|_0^{1-|y|} = 1 \checkmark$$

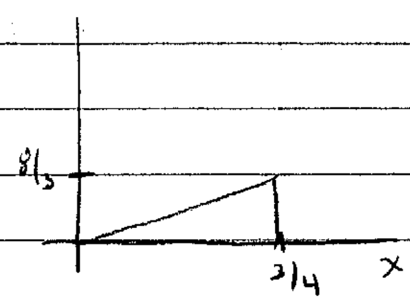
(d) <u>No</u> For example, $f_{X|Y}(x|0) \neq f_{X|Y}(x|0.5)$

(e)

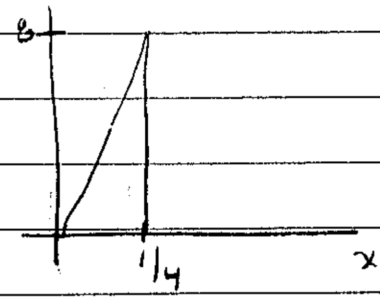
$$f_{X|Y}(x|0.25) = \frac{2x}{(3/4)^2} = \begin{cases} 32/9\ x, & 0 \le x \le 3/4 \\ 0, & \text{else} \end{cases}$$

$$f_{X|Y}(x|0.75) = \frac{2x}{(1/4)^2} = \begin{cases} 32x, & 0 \le x \le 1/4 \\ 0, & \text{else} \end{cases}$$

$f_{X|Y}(x|0.25)$



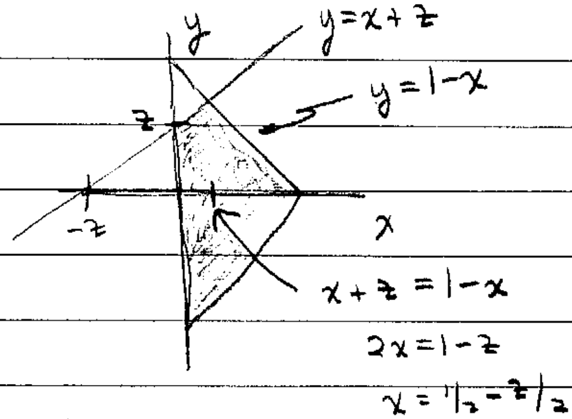
$f_{X|Y}(x|0.75)$



Definitely $Y=0.25$! Lots of ways. Here's one:

$$E[X|Y=0.25] = \int_0^{3/4} 32/9\, x^2\,dx = 32/27 \cdot 27/64 = 1/2$$

$$E[X|Y=0.75] = \int_0^{1/4} 32\, x^2\,dx = 32/3 \cdot 1/64 = 1/6$$



(f) $F_Z(z) = P(Z \le z)$

$= P(Y - X \le z)$

$= P(Y \le z + X)$

$= 1 - \int_0^{1/2 - z/2} \int_{x+z}^{1-x} 3x \, dy \, dx$

2)

(a) <u>no</u> $\rho_{X,Y} \neq 0$ implies X and Y are correlated

$\Longrightarrow$ ~~not independent~~

(b) $Z_1 = 2X + Y$

We know Z_1, a linear combination of jointly Gaussian X and Y, is Gaussian.

$$E[Z_1] \overset{\text{linearity}}{=} 2E[X] + E[Y] = 2 \cdot 1 + 2 = 4$$

$$E[Z_1^2] = E[(2X+Y)^2]$$

$$= 4\underbrace{E[X^2]}_{Var(X)+(E[X])^2 = 10} + 4E[XY] + \underbrace{E[Y^2]}_{Var(Y)+(E[Y])^2 = 20}$$

$$\rho_{XY} = \frac{E[XY] - \cancel{E[X]E[Y]}\ 2}{\sqrt{9 \cdot 16}} \Longrightarrow E[XY] = 12 \cdot 0.4 + 2 = 6.8$$

$$= 40 + 27.2 + 20 = 87.2$$

$$Var(Z_1) = 87.2 - 4^2 = 71.2 \qquad \frac{x-\mu}{\sqrt{2}\sigma} = \frac{3-4}{\sqrt{142.4}} = -1/\sqrt{142.4}$$

$$P(Z_1 > 3) = 1 - P(Z_1 < 3)$$

$$= 1 - \left(\tfrac{1}{2} - \tfrac{1}{2}\,\mathrm{erf}\left(1/\sqrt{142.4}\right)\right)$$

$$= 1/2 + 1/2\,\mathrm{erf}\left(1/\sqrt{142.4}\right)$$

(c)

I claim it goes to $Z=0$ all 3 ways.

$$E[|Z_n-0|^2] = E\left[\left|\frac{2X}{n}+\frac{Y}{n}\right|^2\right]$$

$$= \frac{1}{n^2}\left(4E[X^2]+4E[XY]+E[Y^2]\right)$$

from (b)

$$= \frac{1}{n^2}\cdot 72$$

$$\to 0$$

Thus, $Z_n \xrightarrow{m.s.} 0 \Rightarrow Z_n \xrightarrow{P} 0$ and $Z_n \xrightarrow{d} 0$.

3)

REMOVED FROM EXAM

This does not converge in any way. In particular,

$$f_{Z_n}(x) = \tfrac{1}{2}\delta(x-n) + \tfrac{1}{2}\delta(x+n)$$

$$\Rightarrow F_{Z_n}(x) = \begin{cases} 0, & x < -n \\ 1/2, & -n \le x < n \\ 1, & x \ge n \end{cases}$$

There does not exist a valid CDF $F_Z(x)$ s. th.

$$F_{Z_n}(x) \to F_Z(x)$$

at all continuity points.

Converges in all ways:

For any w, $Z_n(w) \to w$ as $n \to \infty$.

Thus, $Z_n \xrightarrow{a.s.} Z = \omega$.

$$\Rightarrow Z_n \xrightarrow{P} Z = \omega, \; Z_n \xrightarrow{D} \omega$$

For m.s.

$$E[|Z_n - w|^2] = 2\int_0^\infty \tfrac{1}{2} e^{-x} (Z_n(x) - x)^2 \, dx$$

$$= 2\int_0^n \tfrac{1}{2} e^{-x} \cancel{(x-x)^2}^{\,0} dx + 2\int_n^\infty \tfrac{1}{2} e^{-x}(x-n)^2 dx$$

$$\le \int_n^\infty x^2 e^{-x}\, dx$$

[For x large enough, $x^2 e^{-x/2} \le 1$] $\Longrightarrow$ $\lim_{n\to\infty} \int_n^\infty x^2 e^{-x} dx \le \lim_{n\to\infty} \int_n^\infty e^{-x/2} dx = \lim_{n\to\infty} e^{-n/2} \to 0.$

4)

(a)

$$P(Y=k \mid X=p) = P(\underbrace{TTT\ldots T}_{k-1 \text{ times}}\, H) = (1-p)^{k-1} p$$

(Note: geometric random variable with parameter p).

REMOVED FROM EXAM

$$P(Y=k, X=p) = P(Y=k \mid X=p)\, P(X=p)$$

$$= (1-p)^{k-1} p \cdot 1/4$$

$p = 1/8, 1/4, 1/2, 1$

$k = 1, 2, 3, \ldots$

$$P(X=p \mid Y=k) = \frac{P(X=p, Y=k)}{P(Y=k)}$$

$$= \frac{(1-p)^{k-1} p \cdot \cancel{1/4}}{\sum_{v \in \{1/8, 1/4, 1/2, 1\}} (1-v)^{k-1} v \cdot \cancel{1/4}}$$

law of total probability

(b) For $k=2$,

$$P(X=1/8 \mid k=2) = \frac{7/8 \cdot 1/8}{\underbrace{7/8 \cdot 1/8 + 3/4 \cdot 1/4 + 1/2 \cdot 1/2}_{7/64 + 12/64 + 16/64}} = \frac{7/64}{35/64} = 1/5$$

$$P(X=1/4 \mid k=2) = \frac{3/4 \cdot 1/4}{35/64} = 12/35$$

$$P(X=1/2 \mid k=2) = \frac{1/2 \cdot 1/2}{35/64} = 16/35$$

$P(X=1 \mid k=2) = 0$

$X = 1/2$ is most likely

(c) $E[X^2Y] = \sum_p \sum_k p^2 k \, P(X=p, Y=k)$

$= \sum_p p^2 \sum_k k \, P(Y=k \mid X=p) \, \xrightarrow{1/4} \cancel{P(X=p)} \; 1/4$

$= \sum_p p^2 \, E[Y \mid X=p] \cdot 1/4$

$= 1/4 \left((1/8)^2 \cdot 8 + (1/4)^2 \cdot 4 + (1/2)^2 \cdot 2 + 1^2 \cdot 1 \right)$

$= 1/4 \left(1/8 + 1/4 + 1/2 + 1 \right)$

$= 1/4 \left(15/8 \right) = 15/32$

5)

(a) It converges in every way to $X=1$.

$$\frac{(-1)^n}{n} + {}^{\omega}/_{n} + 1 \Rightarrow 1 \quad \text{for all } \omega \in \bar{Q}$$

and $P(\bar{Q}) = 1 \Rightarrow X_n \xrightarrow{a.s.} 1 \Rightarrow X_n \xrightarrow{P} 1 \Rightarrow X_n \xrightarrow{D} 1$

$$E\left[|X_n - 1|^2\right] = \int_0^1 (X_n(x) - 1)^2 p_\omega(x)\,dx \le \int_{\bar{Q}} \left((-1)^n/_n + {}^{\omega}/_{n}\right)^2 p_\omega(x)\,dx$$

$$+ \int_Q 20^2 p_\omega(x)\,dx$$

$|X_n(\omega) - 1| \le 20$ for all ω

$$\le {}^{4}/_{n^2} \int_{\bar{Q}} p_\omega(x)\,dx + 20^2 \underbrace{\cancel{\int_Q p_\omega(x)\,dx}}_{0}$$

$\to 0$ as $n \to \infty$

(b)

Does not converge in any way.

$$\text{Note } f_{X_n(\omega)}(x) = \begin{cases} \frac{1}{n}, & 0 \le x \le n \\ 0, & \text{else} \end{cases}$$

and

$$F_{X_n(\omega)}(x) = \begin{cases} 0, & x \le 0 \\ x/n, & 0 \le x \le n \\ 1, & x > n \end{cases}$$

Any limiting distribution $F_X(x)$ must have some x_0 s.th. $F_X(x_0) = \varepsilon > 0$. (It can't be zero everywhere.)

But $F_{X_n}(x_0) \to 0$ for any x_0

Thus, $X_n \not\xrightarrow{D} X \Rightarrow X_n \not\xrightarrow{P} X \Rightarrow X_n \not\xrightarrow{a.s.} X$

$\Rightarrow X_n \not\xrightarrow{m.s.} X$

(c)

$$F_{X_n}(x) = \begin{cases} 0, & x < 0 \\ 1/2, & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}$$

<u>for</u> <u>all</u> <u>n</u>. Thus $F_{X_n}(x) \to F_X(x) = \begin{cases} 0, & x < 0 \\ 1/2, & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}$

But it does not converge in any other way.
Take any $X(\omega)$ and $\epsilon = 1/4$.

$$P(|X_n(\omega) - X(\omega)| > \epsilon) = P(X_n(\omega) \ne X(\omega))$$

$= 1/2$ (think of $\omega \in \bar{Q}$, where the expansions don't terminate)

for n large

Thus, $X_n \not\xrightarrow{P} X \Rightarrow X_n \not\xrightarrow{a.s.} X$

$\Rightarrow X_n \not\xrightarrow{m.s.} X$

You only had to do a.s. $\omega \in \bar{Q}$, For $X_n(\omega) \not\to X(\omega)$ for any choice of $X(\omega)$, and $P(\bar{Q}) = 1$.