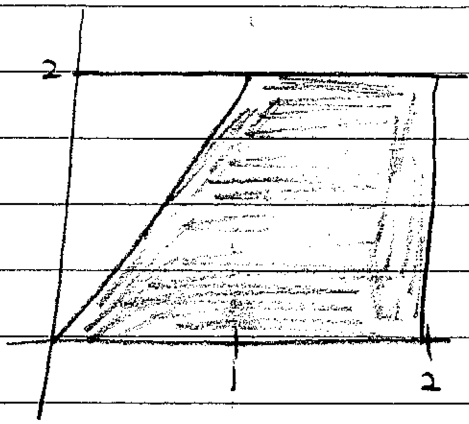
1)



(a) $\int_0^2 \int_{y/2}^2 c(x+y)\,dx\,dy$

$= c\int_0^2 \left(\tfrac{1}{2}x^2 + xy\right)\Big|_{y/2}^{2} dy$

$= c\int_0^2 \left(2 + 2y - y^2/8 - y^2/2\right) dy$

$= c\int_0^2 \left(2 + 2y - 5y^2/8\right) dy = c\left(2y + y^2 - 5y^3/24\right)\Big|_0^2$

$= c\left(4 + 4 - 5/3\right) = c \cdot 19/3 \Rightarrow c = 3/19$

(b) Two parts:

$\underline{0 \le x \le 1}$:

$f_X(x) = \int_0^{2x} 3/19\,(x+y)\,dy = 3/19\left(xy + \tfrac{1}{2}y^2\right)\Big|_0^{2x}$

$= 3/19\left(2x^2 + 2x^2\right) = 12/19\, x^2$

$\underline{1 \le x \le 2}$

$f_X(x) = \int_0^2 3/19\,(x+y)\,dy = 3/19\left(xy + \tfrac{1}{2}y^2\right)\Big|_0^2 = 3/19\,(2x+2)$

So:

$$f_X(x) = \begin{cases} 12/19\, x^2, & 0 \le x \le 1 \\ 6/19\,(x+1), & 1 \le x \le 2 \\ 0, & \text{else} \end{cases}$$

Check:

$12/57\, x^3\Big|_0^1 + 6/19\left(\tfrac{1}{2}x^2 + x\right)\Big|_1^2$

$= 12/57 + 6/19\left(4 - 1/2 - 1\right)$

$= 4/19 + 6/19\,(5/2) = 1$ ✓

(c) $f_{Y|X}(y|x) = \dfrac{f_{X,Y}(x,y)}{f_X(x)}$

$$= \begin{cases} \dfrac{3/\cancel{14}\,(x+y)}{12/\cancel{14}\,x^2} = \dfrac{1}{4}\dfrac{(x+y)}{x^2}, & 0 \le x \le 1,\ 0 \le y \le 2x \\ \dfrac{3/\cancel{14}\,(x+y)}{6/\cancel{14}\,(x+1)} = \dfrac{1}{2}\dfrac{(x+y)}{(x+1)}, & 1 \le x \le 2,\ 0 \le y \le 2 \\ 0, & \text{else} \end{cases}$$

(d)

Look at plot above. If $x=0$, then $Y=0$

$\Rightarrow$ choose $x=0$.

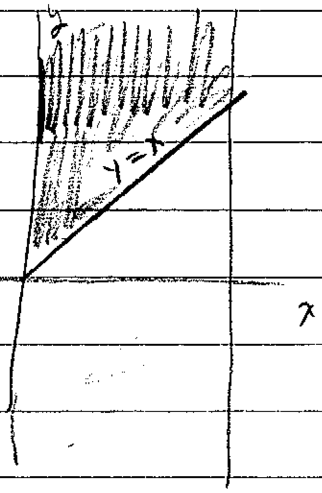
2)

(a) $X > 0,\ Y \in [0, X] \Rightarrow X > Y$
$X < 0,\ Y \in [0, -X] \Rightarrow Y > X$ $\Big\}\ P(Y > X) = P(X < 0) = 1/2$

(b) For any x, $P(Y > x) = 1/2$ (Gaussian distribution is symmetric about the mean), and thus

$P(Y > X) = 1/2$. $\left(P(Y > X) = \int_{-\infty}^{\infty} P(Y > X \mid X = x) f_X(x)\, dx \right)$

(c)



$$P(Y > X) = \int_0^1 \int_x^\infty f_{X,Y}(x,y)\, dy\, dx$$

$$= \int_0^1 \int_x^\infty f_X(x) f_Y(y)\, dy\, dx$$

$$\boxed{P(Y > X) = \int_0^1 \left(\int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy \right) dx}$$

$$= \int_0^1 \left(1 - \int_{-\infty}^{x} f_Y(y)\, dy \right) dx$$

$$= \int_0^1 \left(1 - \left(1/2 + \mathrm{erf}(x) \right) \right) dx = \int_0^1 \left(1/2 - \mathrm{erf}(x) \right) dx$$

(d) • $P((0,2)) = 1 \Rightarrow c = 1/4$

• $P(Y > X) = P(\{\omega : Y(\omega) > X(\omega)\}) = P(\{\omega : \omega^2 > \omega\})$

$= P((1,2))$

$= 3/4$

3)

(a)

$$E[X+Y] = E[X] + E[Y] = 1$$
$$E[X-Y] = E[X] - E[Y] = -3$$
$$2E[X] = -2 \Rightarrow E[X] = -1$$
$$\Rightarrow E[Y] = 2$$

(b) $E[(X+Y)^2] = E[X^2 + 2XY + Y^2] = E[X^2] + 2E[XY] + E[Y^2]$

$E[(X-Y)^2] = E[X^2 - 2XY + Y^2] = E[X^2] - 2E[XY] + E[Y^2]$

subtract $\qquad 4E[XY] = 4$

$$\Rightarrow E[XY] = 1$$

$$\Rightarrow 15 = \underbrace{\sigma^2 + (-1)^2}_{E[X^2]} + 2\cdot 1 + \underbrace{\sigma^2 + 2^2}_{E[Y^2]}$$

$$8 = 2\sigma^2 \Rightarrow \sigma^2 = 4$$

$$\rho_{X,Y} = \frac{E[XY] - E[X]E[Y]}{\sigma_X \sigma_Y} = \frac{1-(-1)(2)}{2\cdot 2} = 3/4$$

(c) • no $\quad$ X & Y jointly Gaussian $\Rightarrow$ all linear combinations are Gaussian

• maybe (same as above)

• yes $\quad$ Apply Fact and * $\quad$ For any α, β:

$$\alpha X + \beta Y = \alpha\left(\tfrac{1}{2}(Z+W)\right) + \beta\left(\tfrac{1}{2}(Z-W)\right)$$
$$= \underbrace{\left(\tfrac{\alpha}{2} + \beta/2\right) Z + \left(\alpha/2 - \beta/2\right) W}$$

but (Z,W) jointly Gaussian $\Rightarrow$ is Gaussian

4)

(a) $X_n(\omega) = (\omega + 1/n)^2$

- $X_1(\omega) = (\omega + 1)^2$

  $X_2(\omega) = (\omega + 1/2)^2$

Note that $X_1(\omega)$ <u>determines</u> $X_2(\omega)$!

$$\omega = \sqrt{X_1(\omega)} - 1$$

$$\Rightarrow X_2(\omega) = \left(\sqrt{X_1(\omega)} - 1 + 1/2\right)^2 = \left(\sqrt{X_1(\omega)} - 1/2\right)^2$$

$$\Rightarrow f_{X_2|X_1}(x_2|x_1) = \delta\left(x_2 - (\sqrt{x_1} - 1/2)^2\right)$$

(b) I claim it goes in <u>all</u> <u>ways</u> to $X(\omega) = \omega^2$

Consider any $\omega \in (0,1)$

$$X_n(\omega) = \omega^2 + 2\omega/n + 1/n^2 \to \omega^2$$

Hence, $X_n(\omega) \to \omega^2$ pointwise $\Rightarrow X_n \xrightarrow{a.s.} X$

$$X_n \xrightarrow{P} X$$

$$X_n \xrightarrow{D} X$$

$$E[|X_n - X|^2] = E[|\cancel{\omega^2} + 2\omega/n + 1/n^2 - \cancel{\omega^2}|^2]$$

$$= E[|2\omega/n + 1/n^2|^2] \le E[(2/n + 1/n^2)^2]$$

$$= 4/n^2 + 4/n^3 + 1/n^4 \to 0 \Rightarrow X_n \xrightarrow{m.s.} X$$

(b)

- $X_1(n) = -w$

Per Exam 2, this flips around the pdf of w. But that pdf is symmetric

$$\Rightarrow f_{X_1}(x) = \begin{cases} 1, & -1/2 \le x \le 1/2 \\ 0, & \text{else} \end{cases}$$

- I claim $X_n \xrightarrow{D} X(w) = w$, but no other way

First, note that $f_{X_n(w)}(x)$ never changes

$\Rightarrow F_{X_n}(x)$ never changes $\Rightarrow F_X(x) \to F_X(x)$, where $X(w) = w$.

Next, consider $X_n \xrightarrow{P} X(w) = w$. Let $\varepsilon = 1/2$.

<u>For n odd:</u>

$$P(|X_n - X| > 1/2) = P(|2w| > 1/2)$$
$$= P(|w| > 1/4)$$
$$= 1/2$$

Thus $P(|X_n - X| > \varepsilon)$ does not go to zero for $\varepsilon = 1/2$

$X_n \cancel{\xrightarrow{P}} X$, $X_n \cancel{\xrightarrow{a.s.}} X$, $X_n \cancel{\xrightarrow{m.s.}} X$

(c)

Looks like it is going to $X=0$.

$$E[|X_n - 0|^2] = E[X_n^2] = 0 + \sum_{i=1}^{n} 1/n^2 \cdot i^2$$

$$= 1/n^2 \sum_{i=1}^{n} i^2$$

$$\geq 1/n^2 \cdot n^2 = 1$$

<u>not</u> in m.s.

<u>For any $\epsilon > 0$:</u>

$$P(|X_n - 0| > \epsilon) = P(|X_n| > \epsilon) = 1/n \to 0 \quad \text{as } n \to \infty$$

$$X_n \xrightarrow{P} 0$$
$$X_n \xrightarrow{d} 0$$