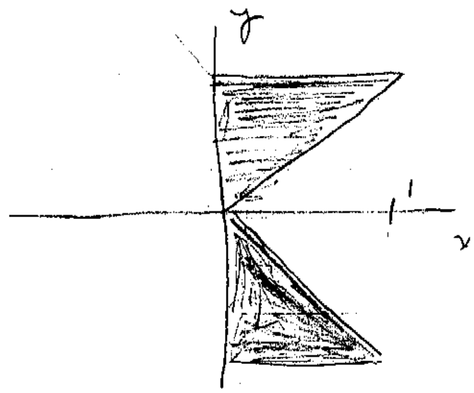
Midterm 2 Solutions

ECE 603 Fall, 2008

1)



a)

$$2\int_0^1\int_x^1 c x^2 y^2 \, dy\, dx$$

$$= 2\int_0^1 \tfrac{1}{3} c x^2 y^3 \Big|_x^1 dx$$

$$= \tfrac{2}{3} c \int_0^1 (x^2 - x^5)\, dx$$

$$= \tfrac{2}{3} c \left(\tfrac{1}{3} - \tfrac{1}{6}\right) = \tfrac{1}{9} c \Rightarrow c = 9$$

(b)

$$P(Y < 1-X) = \tfrac{1}{2} + \int_0^{1/2}\int_x^{1-x} 9x^2y^2\, dy\, dx$$

(c) $$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$$

$$= \int_0^{|y|} 9x^2y^2\, dx$$

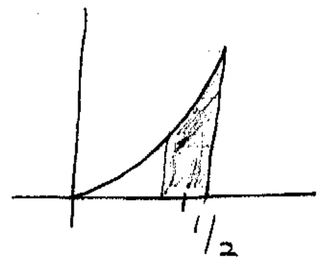
$$= 3x^3y^2 \Big|_0^{|y|}$$

$$= 3|y|^3 y^2 = 3|y|^5 \qquad -1 \le y \le 1$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} 3x^2/|y|^3 & -1 \le y \le 1,\ 0 \le x \le |y| \\ 0, & \text{else} \end{cases}$$

(d)

$$f_{X|Y}(x|0.5) = \begin{cases} 24x^2, & 0 \le x \le 1/2 \\ 0, & \text{else} \end{cases}$$



Choose $x_0 = 0.4$

(e)

So, U is 0 one-half of the time $(Y>0)$ (and never negative); thus

$$F_U(0) = 1/2$$
$$F_U(u) = 0, \quad u < 0.$$

For $0 \le u \le 1$

$$\begin{aligned} F_U(u) &= 1/2 + P(0 \le U \le u) \\ &= 1/2 + P(0 < y^2 \le u \text{ \underline{and} } y < 0) \\ &= 1/2 + P(-\sqrt{u} < y < 0) \\ &= 1/2 + \int_{-\sqrt{u}}^{0} 3(-y)^5 \, dy \\ &= 1/2 - \frac{1}{2} y^6 \Big|_{\sqrt{u}}^{0} \\ &= 1/2 + 1/2\, u^3 \end{aligned}$$

$$\Rightarrow f_U(u) = 1/2\, \delta(u) + \begin{cases} 3/2\, u^2, & 0 \le u \le 1 \\ 0, & \text{else} \end{cases}$$

For $u > 1$, $F_U(u) = 0$

2)

(a) linear combinations of jointly Gaussian X and Y are Gaussian; thus,

$$f_Z(z) = \frac{1}{\sqrt{2\pi\sigma_z^2}} e^{-(z-\mu_z)^2/2\sigma_z^2}$$

$$E[Z] = E[2X-3Y] = 2E[X] - 3E[Y] = 0$$

$$E[Z^2] = E[(2X-3Y)^2]$$

$$= E[4X^2 - 12XY + 9Y^2]$$

$$= 4E[X^2] - 12E[XY] + 9E[Y^2]$$

$$E[XY] = \rho_{XY}\sqrt{E[X^2]E[Y^2]} = 4/3$$

$$\Rightarrow E[Z^2] = 16 - 16 + 36$$

$$= 36$$

$$P(Z>3) = 1 - P(Z \le 3)$$

$$\frac{x-\mu}{\sigma} = 1/2 \quad \overset{\downarrow}{=} 1 - (1/2 + \text{erf}(1/2))$$

$$= 1/2 - \text{erf}(1/2)$$

(b) independent $\Rightarrow$ uncorrelated $\Rightarrow \rho_{X,Y} = 0$

$$E[X^2Y^2] \overset{\text{indep}}{=} E[X^2]E[Y^2] = 4$$

$$E[X^2+Y^2] = E(X^2) + E(Y^2) = 5$$

$$\Rightarrow \begin{array}{l} \sigma_1^2\sigma_2^2 = 4 \\ \sigma_1^2 + \sigma_2^2 = 5 \end{array} \Rightarrow \sigma_1^2 + 4/\sigma_1^2 = 5 \Rightarrow (4, 1) \text{ and } (1, 4) \text{ for } (\sigma_1^2, \sigma_2^2)$$

Lots of ways to do this. I let $Y = \omega$, and note $X = Y^2$.

$$F_Y(y) = P(Y \le y)$$
$$= P(\omega \le y)$$
$$= P(0 \le \omega \le y), \qquad 0 \le y \le 1$$
$$= P(\omega \in (0, y))$$
$$= \begin{cases} y^2/2, & y < 1/4 \\ 17/32 & y = 1/4 \\ \frac{1}{2} + y^2/2, & y > 1/4 \end{cases}$$

$$f_Y(y) = \begin{cases} y, & 0 \le y \le 1 \\ 0, & \text{else} \end{cases}$$
$$+ \ 1/2 \, \delta(y - 1/4)$$

Now $E[X] = E[Y^2] = 1/2 \cdot (1/4)^2 + \int_0^1 y^3 \, dy$

$$= 1/32 + 1/4$$
$$= 9/32$$

4)

$$E[X] = E[\cos \Theta] = \int_0^{2\pi} 1/2\pi \cos \theta \, d\theta$$

$$= 1/2\pi (+\sin \theta) \Big|_0^{2\pi} = 0$$

$$E[Y] = E[\sin \Theta] = \int_0^{2\pi} 1/2\pi \sin \theta \, d\theta$$

$$= 1/2\pi (-\cos \theta) \Big|_0^{2\pi} = 0$$

$$E[XY] = E[\sin \Theta \cos \Theta] = \int_0^{2\pi} 1/2\pi \sin \theta \cos \theta \, d\theta$$

$$= 1/4\pi \int_0^{2\pi} \sin 2\theta \, d\theta$$

$$= 1/4\pi \cdot 1/2 (-\cos 2\theta) \Big|_0^{2\pi}$$

$$= 0$$

$\Rightarrow E[XY] = E[X]E[Y]$ <u>uncorrelated</u>

Are they independent?

$$f_{X|Y}(x \mid y=0) = 1/2\, \delta(x-1) + 1/2\, \delta(x+1)$$

$$f_{X|Y}(x \mid y=1) = \delta(x)$$

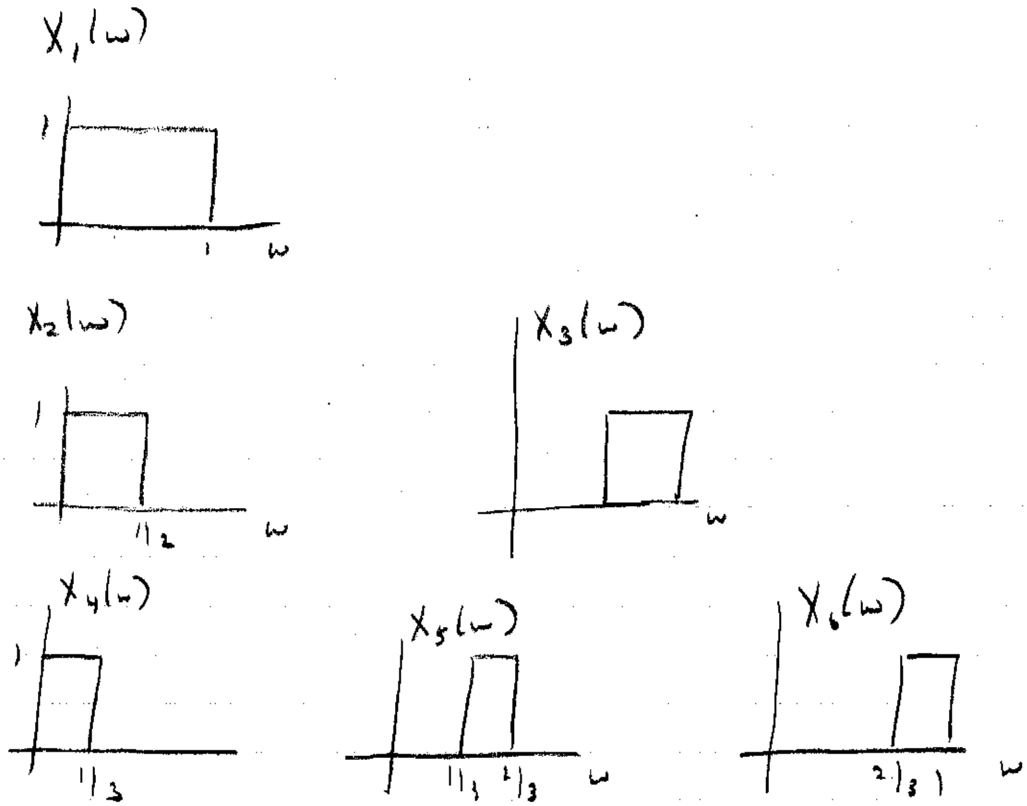
$$f_{X|Y}(x \mid y=0) \neq f_{X|Y}(x \mid y=1)$$

<u>not independent</u>

(b) $(\Omega, \mathcal{A}, P)$ $\quad \Omega = [0,1]$

$\mathcal{A} = \mathcal{B}$

$P((a,b)) = b - a$



⋮

From class $X_n \cancel{\xrightarrow{a.s.}} 0$

m.s.

$$E[|X_n - X|^2] = 1^2 \cdot P(X_n(\omega) = 1) + 0^2 \cdot P(X_n(\omega) = 0)$$

$$\to 0 \quad \text{as} \quad n \to \infty$$

5) (a)

$$X_n(\omega) \to X = 1$$

How? in every way

Consider any $\omega \in (0,1)$ (a set of probability 1)

$$X_n(\omega) = 1 - \omega/n \to 0$$

Hence

$$X_n \xrightarrow{a.s} X \Rightarrow X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{D} X$$

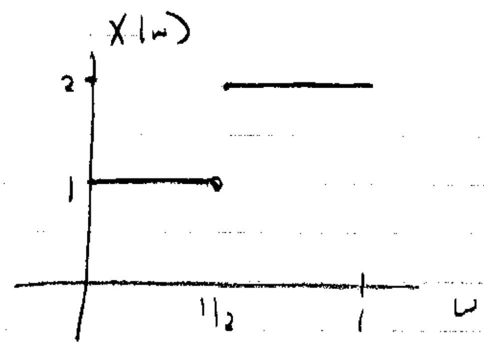
m.s.

$$E[|X_n(\omega) - 1|^2] = E[|\omega/n|^2] \le E[1/n^2] = 1/n^2 \to 0$$

$$\Rightarrow X_n \xrightarrow{m.s.} X$$

(b)

$$X_n(\omega) \to X(\omega)$$



How? In every way.

Consider any $\omega \in (0,1)$ (a set of probability 1)

$$|X_n(\omega) - X(\omega)| = 1/n \to 0 \quad \text{as } n \to \infty$$

$$\Rightarrow X_n \xrightarrow{a.s.} X \Rightarrow X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{D} X$$

m.s.

$$E[|X_n(\omega) - X(\omega)|^2] = 1/n^2 \to 0 \quad \text{as } n \to 0$$

$$\Rightarrow X_n \xrightarrow{m.s.} X$$

(c) <u>does</u> <u>not</u> <u>converge</u>. I will disprove almost sure

Let $A = \{1/2\}$, a set of probability $1/2$.
For $w \in A$

$$X_n(w) = -1, +1, -1, +1, -1, +1, \ldots$$

which is not convergent. Hence, there cannot exist a set of probability one s.th.

$$X_n(w) \to X(w) \text{ on that set.}$$