1) $0 \le x \le 3$

$$P(w \le x) = \frac{\pi x^2}{\pi 3^2} = x^2/9$$

$$f_w(x) = \frac{dP(w \le x)}{dx} = \begin{cases} 2x/9, & 0 \le x \le 3 \\ 0, & \text{else} \end{cases}$$

$\Omega = [0, 3]$

$\mathcal{A} = \mathcal{B}$ restricted to $[0,3]$

$$P((a,b)) = b^2/9 - a^2/9 = (b^2 - a^2)/9$$

(and all of the other sets in $\mathcal{A}$ are generated from sets of the form (a,b))

2)

(a) $$P(m=5) = \sum_{i=1}^{6} P(m=5 \mid N=i)\, P(N=i)$$

$$= P(m=5 \mid N=5)\, 1/6 + P(m=5 \mid N=6)\, 1/6$$

$$= 1/6 \left((1/2)^5 + \binom{6}{5} (1/2)^5 (1/2)^1 \right)$$

$$= 1/6 \left(1/32 + 6/64 \right)$$

$$= 1/48$$

(b) $$P(N=6 \mid m=5) = \frac{P(m=5 \mid N=6)\, P(N=6)}{P(m=5)} = \frac{6/64 \cdot 1/6}{1/48}$$

$$= 3/4$$

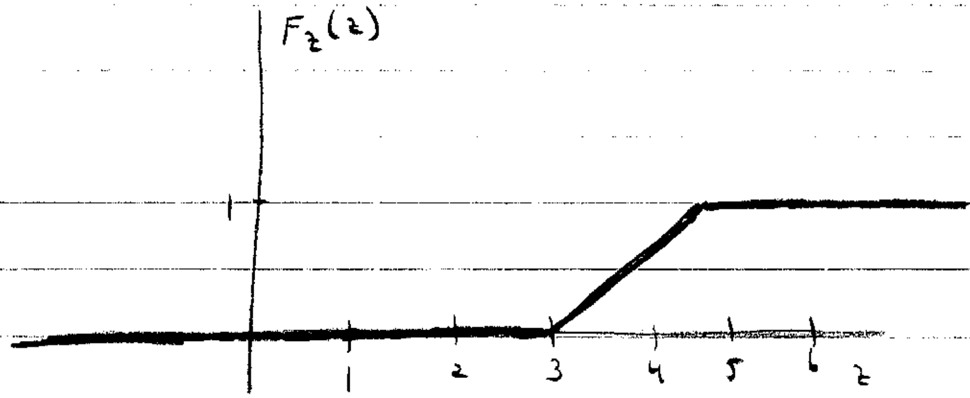
3) (a)

Note - X is uniform between $1/2$ and 2.

- Y is $5/2$ with probability one.

Go to Y!

(b) Since $Y = 5/2$, the cdf is just a shifted version of X.



4) I claim it converges to $X=0$ in all four ways. First, for any irrational number in $(0,1)$

$$X_n(\omega) = \omega^2/n \longrightarrow 0 \quad \text{as } n \to \infty$$

Since $A = \{\omega : \omega \text{ irrational}\}$ has $P(A) = 1$

$$X_n \xrightarrow{a.s} 0 \Rightarrow X_n \xrightarrow{P} 0 \Rightarrow X_n \xrightarrow{D} 0$$

For mean square,

$$E[|X_n - X|^2] = E[|X_n|^2] = E[|X_n|^2 \mid \omega \text{ irrational}]\,\overbrace{P(\omega \text{ irrational})}^{1} + E[|X_n|^2 \mid \omega \text{ rational}]\,\overbrace{P(\omega \text{ rational})}^{0}$$

$$= E[w^4/n^2] = 1/n^2 \, E[w^4] = 1/5n^2 \to 0 \quad \text{as } n \to \infty$$

$$\Longrightarrow X_n \xrightarrow{m.s.} 0$$

5) (a)

$Y[n]$: number of heads in n flips of a coin

$$P(Y[n]=j) = \binom{n}{j} (1/2)^j (1/2)^{n-j}$$

$$= \binom{n}{j} 1/2^n$$

(b)

$$m_Y[n] = E[Y[n]] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} 1/2 = n/2$$

(c)

$$R_Y[m,n] = E[Y[m]Y[n]]$$

$$= E\left[\sum_{i=1}^{m} X_i \sum_{j=1}^{n} X_j\right]$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} E[X_i X_j] = \begin{cases} 1/4, & i \neq j \\ 1/2, & i = j \end{cases}$$

$$= 1/2 \min(n,m) + 1/4 (nm - \min(m,n))$$

(d) No! If it converges, it must converge (at least) in distribution to some $F_X(x)$ for which $P(X \leq x_0) > 0.5$ for some x_0. But the probability that an infinite sequence has less than x_0 1's is zero; thus,

$$X_n \not\xrightarrow{p} X$$

$$\Longrightarrow \quad X_n \not\xrightarrow{a.s.} X, \; X_n \not\xrightarrow{r} X, \; X_n \not\xrightarrow{m.s.} X$$

6) (a)

$$m_z(t) = E\left[X(t)\cos(2\pi f_c t) + Y(t)\sin(2\pi f_c t)\right]$$

$$= \cancel{E[X(t)]}^{\,0}\cos(2\pi f_c t) + \cancel{E[Y(t)]}^{\,0}\sin(2\pi f_c t)$$

$$= 0$$

(b) $R_z(t_1, t_2) = E[Z(t_1)Z(t_2)]$

$$= E\left[(X(t_1)\cos(2\pi 20 t_1) + Y(t_1)\sin(2\pi 20 t_1))(X(t_2)\cos(2\pi 20 t_2) + Y(t_2)\sin(2\pi 20 t_2))\right]$$

$$= \cos(2\pi 20 t_1)\cos(2\pi 20 t_2)\,E[X(t_1)X(t_2)]$$

$$+ \cos(2\pi 20 t_1)\sin(2\pi 20 t_2)\,\cancel{E[X(t_1)Y(t_2)]}^{\,0}$$

$$+ \sin(2\pi 20 t_1)\cos(2\pi 20 t_2)\,\cancel{E[Y(t_1)X(t_2)]}^{\,0}$$

$$+ \sin(2\pi 20 t_1)\sin(2\pi 20 t_1)\,E[Y(t_1)Y(t_2)]$$

$$= \frac{5\sin(3\pi(t_1 - t_2))}{3\pi(t_1 - t_2)}\left(\cos(2\pi 20 t_1)\cos(2\pi 20 t_2) + \sin(2\pi 20 t_1)\sin(2\pi 20 t_2)\right)$$

$$= \frac{5\sin(3\pi(t_1-t_2))}{3\pi(t_1-t_2)} \quad \frac{1}{2}\cos(2\pi 20(t_1-t_2))$$

$$= 5\,\frac{\sin(3\pi(t_1-t_2))}{3\pi(t_1-t_2)}\cos(2\pi 20(t_1-t_2))$$

(c) Fix t:

Since $X(t)$ and $Y(t)$ are Gaussian & independent

$\Rightarrow X(t)$ and $Y(t)$ are jointly Gaussian.

$\Rightarrow Z(t)$ is Gaussian

$$\mu = E[Z(t)] = 0$$

$$\sigma^2 = R_Z(0) = 5$$

$$\Rightarrow f_{Z(t)}(x) = \frac{1}{\sqrt{\pi 10}} e^{-x^2/10}$$

(d) I will choose $t_1 = 0$, $t_2 = 1/3$. Then

$R_X(t_1-t_2) = 0 \Rightarrow X(t_1), X(t_2)$ independent (jointly Gaussian & uncorrelated)

$R_Y(t_1-t_2) = 0 \Rightarrow Y(t_1), Y(t_2)$ independent

$\Rightarrow Z(0), Z(1/3)$ are each Gaussian and independent.

$$\Rightarrow f_{Z(0),Z(1/3)}(x_1, x_2) = f_{Z(0)}(x_1)\, f_{Z(1/3)}(x_2) = \frac{1}{\sqrt{\pi 10}} e^{-x_1^2/10} \frac{1}{\sqrt{\pi 10}} e^{-x_2^2/10}$$

$$= \frac{1}{10\pi} e^{-(x_1^2+x_2^2)/10}$$