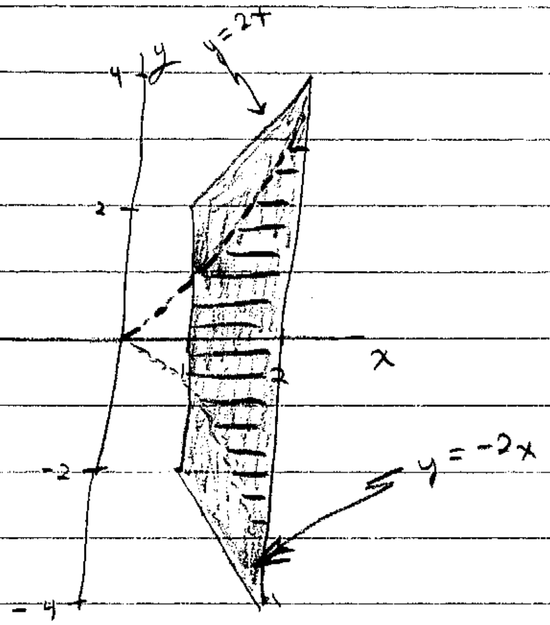
Midterm #2 Solutions

ECE 603 Fall, 2015

1)

(a)



$$\int_1^2 \int_{-2x}^{2x} c\, dy\, dx = \int_1^2 c \cdot 4x\, dx = c \cdot 2x^2 \Big|_1^2 = c \cdot 6 \Rightarrow c = 1/6$$

(Note: Also could do $A = \frac{b_1 + b_2}{2} \cdot h = \frac{4+8}{2} \cdot 1 = 6$)

(b)

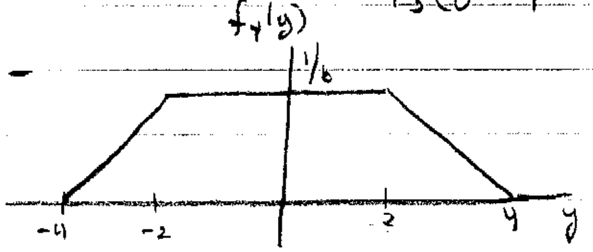
$$f_Y(y) = \begin{cases} \int_{-y/2}^{2} 1/6\, dx = (2 + y/2) \cdot 1/6, & -4 \le y \le -2 \\ \int_1^2 1/6\, dx = 1/6, & -2 \le y \le 2 \\ \int_{y/2}^{2} 1/6\, dx = (2 - y/2) \cdot 1/6, & 2 \le y \le 4 \\ 0, & \text{else} \end{cases}$$

Check: $\int_{-\infty}^{\infty} f_Y(y)\, dy = 2 \int_2^4 (2 - y/2) \cdot 1/6\, dy + \int_{-2}^{2} 1/6\, dy$

$= 1/3 \left(2y - y^2/4\right)\Big|_2^4 dy + 2/3$

$= 1/3 (8 - 4 - 4 + 1) + 2/3 = 1$

-or-



area = 1.

(c) $P(|Y| < X^2) = \int_1^2 \int_{-x^2}^{x^2} 1/6 \, dy \, dx$

$= \int_1^2 x^2/3 \, dx$

$= x^3/9 \Big|_{x=1}^{x=2}$

$= 7/9$

(d) $f_{X|Y}(x|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)}$

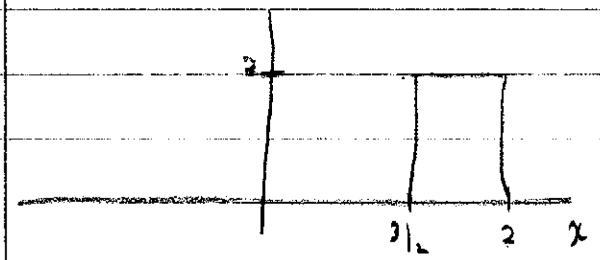
$$= \begin{cases} \dfrac{1}{2 + y/2}, & -4 \le y \le -2, \ -y/2 \le x \le 2 \\ 1, & -2 \le y \le 2, \ 1 \le x \le 2 \\ \dfrac{1}{2 - y/2}, & 2 \le y \le 4, \ y/2 \le x \le 2 \end{cases}$$

(<u>Note</u>: Integrates to 1 for any y!)

(e) <u>No</u>! $f_{X|Y}(x|y)$ depends on y.

(f)

$f_{X|Y}(x|3.0) = \dfrac{1}{2 - 3/2} = 2, \quad 3/2 \le x \le 2$



Given $Y = 3.0$, X is uniform on $[3/2, 2]$.

Choose $\hat{X} = 7/4$. (expected value)

2)

(a)

$X \sim N(1, 16)$

$$\Rightarrow W \sim N(6, 16) \Rightarrow f_W(w) = \frac{1}{\sqrt{2\pi 16}} e^{-(w-6)^2/32}$$

(b)

$E[V] = 3E[X] + 2 = 5$

$Var[V] = 3^2 Var(X) = 9 \cdot 16 = 144$

$$f_V(v) = \frac{1}{\sqrt{2\pi 144}} e^{-(v-5)^2/288}$$

(c) Z is a linear combination of jointly Gaussian X and Y $\Rightarrow$ Z is Gaussian.

$E[Z] = E[X] - 3E[Y] = 4$

$E[Z^2] = E[(X-3Y)^2] = E[X^2] - 6E[XY] + 9E[Y^2]$

$E[X^2] = Var(X) + (E[X])^2 = 16 + 1 = 17$

$E[Y^2] = Var(Y) + (E[Y])^2 = 9 + 1 = 10$

$$\frac{E[XY] - E[X]E[Y]}{\sqrt{Var(X)Var(Y)}} = 1/3 \Rightarrow E[XY] = 4 + (1)\cdot(-1) = 3$$

$E[Z^2] = 17 - 18 + 90 = 89 \quad Var(Z) = 89 - (4)^2 = 73$

$$f_Z(z) = \frac{1}{\sqrt{2\pi 73}} e^{-(z-4)^2/2\cdot 73}$$

3)

(a) Converges in all ways to $X=0$.

For any ω, $\omega^3/\sqrt{n} \to 0$ as $n\to\infty$; thus, $X_n \xrightarrow{a.s.} X$

$$\Rightarrow X_n \xrightarrow{P} X$$

$$\Rightarrow X_n \xrightarrow{D} X$$

$$E\left[\left|\omega^3/\sqrt{n} - 0\right|^2\right] = E\left[\omega^6/n\right] = 1/n \int_0^1 \omega^6 \, d\omega = 1/7n \to 0$$

$$\Rightarrow X_n \xrightarrow{m.s.} X$$

(b)

Converges in all ways to $X=0$.

For any irrational ω, $\omega/n \to 0$ as $n\to\infty$. Since $P(\overline{\mathbb{Q}})=1$ for this $P(\cdot)$, $X_n \xrightarrow{a.s.} X=0$

$$\Rightarrow X_n \xrightarrow{P} X=0$$

$$\Rightarrow X_n \xrightarrow{D} X=0$$

$$|X_n - X|^2 = \begin{cases} \omega^2/n^2, & \omega \text{ irrational} \\ 1, & \omega \text{ rational} \end{cases}$$

$$\Rightarrow E[|X_n - X|^2] = \omega^2/n^2 \cdot 1 + 1 \cdot 0 = \omega^2/n^2 \to 0 \text{ as } n\to\infty$$

$$X_n \xrightarrow{m.s.} X = 0$$

(c) It doesn't go to $X=0$ in any way.

Since $P(\omega = 1/4) = 1/2$ $P(X_n = 1) \geq 1/2$ for any n

Hence $F_{X_n}(1/2) = P(X \leq 1/2) \leq 1/2$

but $F_X(1/2) = 1$; thus, $F_{X_n}(x) \not\to F_X(x)$ at continuity point $x = 1/2$

~~Thus, $X_n \xrightarrow{D} X$ (and, thus, in any other way)~~

(part (d) is on the next page)

~~(e) $X_n(\omega)$ is uniform on $[0,1]$ for any n.~~

Thus, $X_n(\omega) \xrightarrow{D} X$ uniform on $[0,1)$

But it does not converge any other way.

Consider $\varepsilon = 0.01$

$$P(|X_{n+1} - X_n| < 2\varepsilon) = P(1/2 - \varepsilon \leq \omega \leq 1/2 + \omega) = 2\varepsilon.$$

for any n.

Thus, there does not exist X s.th.

$\lim_{\substack{n\to\infty \\ n \text{ odd}}} P(|X_n - X| < \varepsilon) \to 0$ — thus $X_n \not\xrightarrow{P} X$

and

$\lim_{\substack{n\to\infty \\ n \text{ even}}} P(|X_n - X| < \varepsilon) \to 0$ $\Rightarrow X_n \not\xrightarrow{a.s.} X$

$\Rightarrow X_n \not\xrightarrow{m} X$

(d) $X_n(\omega) \to \omega$ in all ways

For any ω, $|X_n(\omega) - \omega| \leq 1/n$

$\Rightarrow X_n \xrightarrow{a.s.} X$

$\Rightarrow X_n \xrightarrow{P} X$

$\Rightarrow X_n \xrightarrow{D} X$

$E[|X_n(\omega)-\omega|^2] \leq E\{1/n^2\} = 1/n^2 \to 0$ as $n \to \infty$

$\Rightarrow X_n \xrightarrow{m.s.} X$

4) (a)

- $\int_0^5 c\,dx = 5c \Rightarrow c = 1/5$

- X is uniform on $[0,5]$. $X+2$ is uniform on $[2,7]$

$$f_{X+2}(x) = \begin{cases} 1/5, & 2 \le x \le 7 \\ 0, & \text{else} \end{cases}$$

- $-X$ is uniform on $[-5,0]$

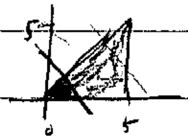
$$f_{-X}(x) = \begin{cases} 1/5, & -5 \le x \le 0 \\ 0, & \text{else} \end{cases}$$

- $X+X = 2X$ is uniform on $[0,10]$

$$f_X(x) = \begin{cases} 1/10, & 0 \le x \le 10 \\ 0, & \text{else} \end{cases}$$

(b)

$$f_{X,Y}(x,y) = \begin{cases} 1/x \cdot 1/5, & 0 \le x \le 5,\ 0 \le y \le x \\ 0, & \text{else} \end{cases}$$



$$f_Y(y) = \int_y^5 1/5x \, dx = 1/5 \ln x \Big|_y^5$$

$$= \begin{cases} 1/5(\ln 5 - \ln y), & 0 \le y \le 5 \\ 0, & \text{else} \end{cases}$$

Check:

$$1/5 \int_0^5 (\ln 5 - \ln y)\,dy$$

$$= \ln 5 - \frac{1}{5}\left(y(\ln y - 1)\right)\Big|_0^5$$

$$= \ln 5 - [(\ln 5 - 1)]$$

$$= 1 \ \checkmark$$

(c) Method 1: (using $f_{X,Y}(x,y)$)

$$F_Z(z) = P(Z \le z)$$

$$= P(X+Y \le z)$$

$$\overset{z \le 5}{=} \int_0^{z/2} \int_y^{z-y} 1/x \cdot 1/5 \, dx\,dy = \int_0^{z/2} \left(1/5 \ln x\right)\Big|_y^{z-y} dy$$

$$= \int_0^{z/2} \left(\tfrac{1}{5} \ln(z-y) - \tfrac{1}{5} \ln y \right) dy$$

$$f_Z(z) = \tfrac{d}{dz} F_Z(z)$$

$$= \int_0^{z/2} \tfrac{1}{5} \frac{1}{z-y}\, dy + \tfrac{1}{2}\left(\tfrac{1}{5}\ln(\tfrac{z}{2}) - \tfrac{1}{5}\ln \tfrac{z}{2}\right) \rightarrow 0$$

$$= -\tfrac{1}{5} \ln(z-y) \Big|_0^{z/2}$$

$$= \tfrac{1}{5}\left(\ln z - \ln \tfrac{z}{2}\right) = \tfrac{\ln 2}{5}$$

For $z \geq 5$:

$$F_Z(z) = \int_{z/2}^{5} \int_{z-x}^{x} \tfrac{1}{5x}\, dy\, dx$$

$$= \int_{z/2}^{5} \left(\frac{2x - z}{5x}\right) dx$$

$$= \int_{z/2}^{5} \left(\tfrac{2}{5} - \tfrac{z}{5x}\right) dx$$

$$f_Z(z) = \tfrac{d}{dz} F_Z(z)$$

$$= +\tfrac{1}{2}\left(\tfrac{2}{5} - \tfrac{2}{5}\right) \rightarrow 0 + \int_{z/2}^{5} \left(+\tfrac{1}{5x}\right) dx$$

$$= +\tfrac{1}{5} \ln x \Big|_{z/2}^{5}$$

$$= -\tfrac{1}{5} \ln \tfrac{z}{2} + \tfrac{1}{5} \ln 5$$

$$= -\tfrac{\ln z}{5} + \tfrac{\ln 2}{5} + \tfrac{1}{5} \ln 5$$

$$= \tfrac{1}{5}\left(\ln 10 - \ln z\right)$$

Thus,

$$f_Z(z) = \begin{cases} \ln 2/5, & 0 \le z \le 5 \\ (\ln 10 - \ln z)/5, & 5 \le z \le 10 \\ 0, & \text{else} \end{cases}$$

<u>Check:</u> $\int_{-\infty}^{\infty} f_Z(z)\,dz = \int_0^5 \ln 2/5 \; dz + \int_5^{10} (\ln 10 - \ln z)/5 \; dz$

$$= \ln 2 + \ln 10 - \frac{1}{5}\int_5^{10} \ln z \, dz$$

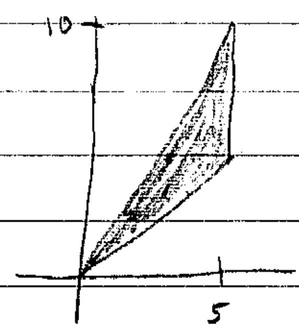
$$= \ln 2 + \ln 10 - \frac{1}{5}\left(z(\ln z - 1)\right)\Big|_5^{10}$$

$$= \ln 2 + \ln 10 - \frac{1}{5}\left(10 \ln 10 - 10\right) + \frac{1}{5}\left(5 \ln 5 - 5\right)$$

$$= \ln 2 + \ln 10 - 2 \ln 10 + 2 + \ln 5 - 1$$

$$= 1 \quad \checkmark$$

<u>Method 2</u> (using $f_{X,Z}(x,z)$)



$$f_{Z|X}(z|x) = \begin{cases} 1/x, & x \le z \le 2x \\ 0, & \text{else} \end{cases}$$

$$\Rightarrow f_{X,Z}(x,z) = \begin{cases} 1/5x, & x \le z \le 2x, \; 0 \le x \le 5 \\ 0, & \text{else} \end{cases}$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Z}(x,z)\,dx = \begin{cases} \int_{z/2}^{z} 1/5x \, dx = 1/5 \ln x \Big|_{z/2}^{z} = 1/5 \ln 2, & 0 \le z \le 5 \\ \int_{z/2}^{5} 1/5x \, dx = 1/5 \ln x \Big|_{z/2}^{5} = 1/5 (\ln 5 - \ln(z/2)) \\ \quad = 1/5 (\ln 10 - \ln z), & 5 \le z \le 10 \end{cases}$$