Midterm 2 Solutions

ECE 603 Fall, 2006

1)

(a)

$$f_X(x) = \frac{1}{\sqrt{2\pi 3}} e^{-(x-3)^2/2\cdot 3}$$

$X \sim N(3, 3)$

$E[X] = 3$
$Var[X] = 3$
$E[X^2] = Var[X] + (E[X])^2 = 12$

Removed from exam.

$$P(X^2 \le 2) = P(-\sqrt{2} \le X \le \sqrt{2})$$
$$= P(X \le \sqrt{2}) - P(X \le -\sqrt{2})$$
$$= \frac{1}{2} - \operatorname{erf}\left(\frac{3-\sqrt{2}}{\sqrt{3}}\right) - \left(\frac{1}{2} - \operatorname{erf}\left(\frac{3+\sqrt{2}}{\sqrt{3}}\right)\right)$$
$$= \operatorname{erf}\left(\frac{3+\sqrt{2}}{\sqrt{3}}\right) - \operatorname{erf}\left(\frac{3-\sqrt{2}}{\sqrt{3}}\right)$$

(b) $\underline{y<0}$: $F_Y(y) = P(Y \le y) = 0$

$\underline{y=0}$: $F_Y(y) = P(Y \le 0) = P(X \le 0) = \frac{1}{2} - \operatorname{erf}(\sqrt{3})$

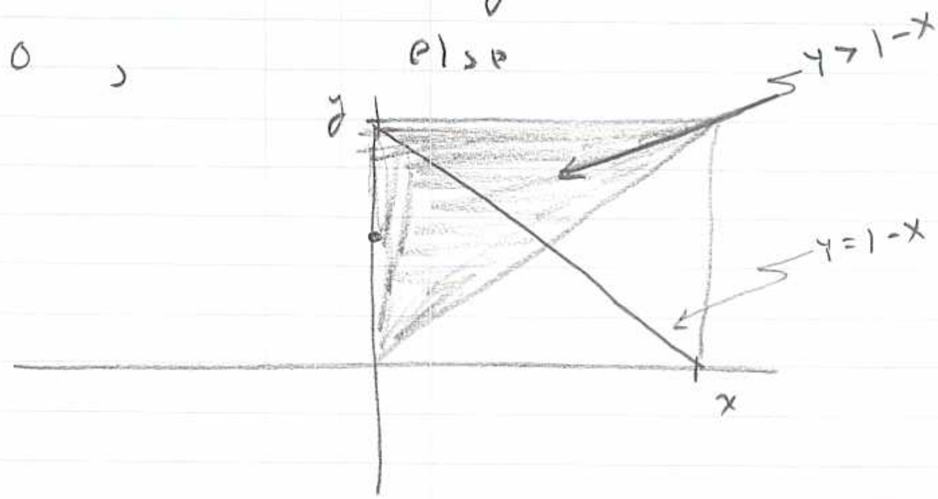
$\underline{y \ge 0}$: $F_Y(y) = P(Y \le y)$

$$= P(3X \le y)$$
$$= P(X \le y/3)$$
$$= \begin{cases} \frac{1}{2} - \operatorname{erf}\left(\frac{3-y/3}{\sqrt{3}}\right) & y \le 9 \\ \frac{1}{2} + \operatorname{erf}\left(\frac{y/3-3}{\sqrt{3}}\right) & y > 9 \end{cases}$$

2)

(a) $f_{X,Y}(x,y) = f_{Y|X}(y|x)\, f_X(x)$

$$= \begin{cases} \frac{1}{1-x} \cdot 1 , & x \le y \le 1,\ 0 \le x \le 1 \\ 0 , & \text{else} \end{cases}$$



$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$$

$$= \int_0^y 1/(1-x)\, dx$$

$$= -\ln(1-x) \Big|_0^y$$

$$= -\ln(1-y), \qquad 0 \le y \le 1$$

(b) (see figure)

$$P(Y > X) = \iint_{\{y > x\}} f_{X,Y}(x,y)\, dy\, dx$$

$$= \int_0^{1/2} \int_{1-x}^1 \frac{1}{1-x}\, dy\, dx + \int_{1/2}^1 \int_x^1 \frac{1}{1-x}\, dy\, dx$$

$$= \int_0^{1/2} \frac{y}{1-x} \Big|_{1-x}^1 dx + \int_{1/2}^1 \frac{y}{1-x} \Big|_x^1 dy$$

$$= \int_0^{1/2} \left(1/_{1-x} - 1 \right) dx + \int_{1/2}^1 \cancel{\left(1/_{1-x} - x/_{1-x} \right)} dx \rightarrow 1$$

$$= \int_0^{1/2} 1/_{1-x}\, dx = -\ln(1-x) \Big|_0^{1/2} = -\ln 1/2 = \ln 2$$

3) (a)

$y \backslash x$	0	1	2	3
0	0	0	1/8	0
1	0	1/4	1/8	0
2	1/8	1/4	0	0
3	1/8	0	0	0

Write down outcomes and P( ):

	X	Y	P( )
TTT	0	3	1/8
TTH	0	2	1/8
THT	1	2	1/8
THH	1	1	1/8
HTT	1	2	1/8
HTH	1	1	1/8
HHT	2	1	1/8
HHH	2	0	1/8

(b)

$$\text{cov}(X,Y) = E[XY] - E[X]E[Y]$$

$$= \left(1/4 \cdot 1 \cdot 1 + 1/4 \cdot 1 \cdot 2 + 1/8 \cdot 1 \cdot 2\right)$$

$$- 1 \cdot 3/2$$

$$= 1/4 + 1/2 + 1/4 - 3/2 = -1/2$$

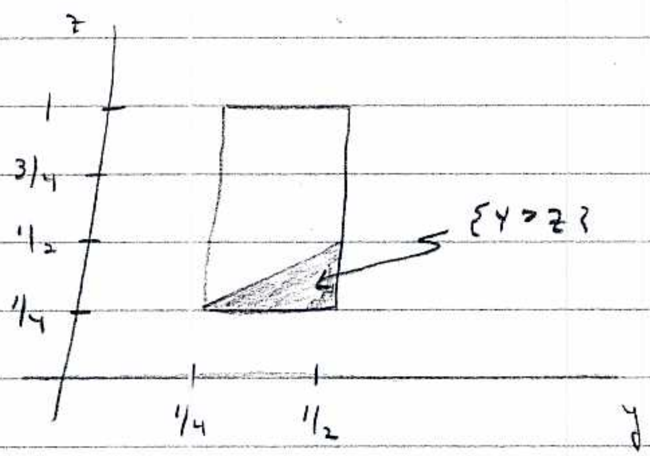
4) (a) I would certainly buy Z stock - lots of reasons (including part (b) below), but an easy one is:

$$E[Z] = 5/8 > 3/8 = E[Y]$$

(b) Since Y and Z are independent,

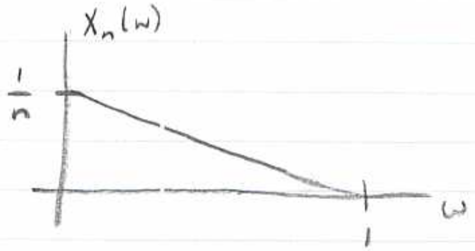
$$f_{Y,Z}(y,z) = f_Y(y)\, f_Z(z)$$

$$= \begin{cases} 16/3, & 1/4 \le y \le 1/2,\ 1/4 \le z \le 1 \\ 0, & \text{else} \end{cases}$$



$$P(Y > Z) = \underbrace{1/4 \cdot 1/4 \cdot 1/2}_{\text{area of triangle}} \cdot 16/3 = 1/6$$

5) a)



Looks like it is going to $X = 0$.

Check a.s.

Consider any $w \in [0,1]$. Choose any $\varepsilon > 0$.
For $N_0 = \lceil 1/\varepsilon \rceil$, $X_n(w) < 1/N_0 < \varepsilon \Rightarrow X_n(w) \to 0$.
Hence,

$$X_n \to X \text{ pointwise}$$
$$\Rightarrow X_n \to X \text{ a.s.}$$
$$\Rightarrow X_n \xrightarrow{P} X$$
$$\Rightarrow X_n \xrightarrow{D} X$$

For m.s. $E[|X_n - X|^2] = E[X_n^2]$

$$= \int_0^1 \left(\tfrac{1}{n}(1-w)\right)^2 dw$$
$$= \frac{1}{n^2} \int_0^1 (1-w)^2 \, dw \to 0 \quad \text{as } n \to \infty$$

Thus, $X_n \xrightarrow{m.s.} X$

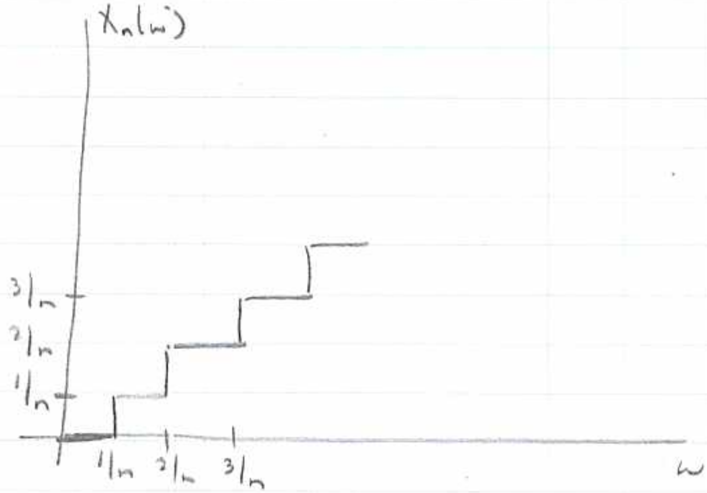
(b)

This does not converge because w^n "grows" on $[1,2]$. To show something rigorous, consider a.s. and any $w \in A = [3/2, 2]$. Now, for large enough n, $X_{n+1}(w) - X_n(w) > 1$.

$\Rightarrow X_n(w)$ does not converge for any $w \in A$
Since $P(A) = 1/4$, $X_n(w)$ does not converge a.s.

(c)

First, recognize what is going on:



I claim $X_n(\omega) \to X(\omega) = \omega$

The key observation is to note that, for any ω,

$$|X_n(\omega) - X(\omega)| < 1/n$$

Thus, consider any $\omega \in [0,1]$

$$\lim_{n\to\infty} |X_n(\omega) - X(\omega)| = 0$$

and $X_n \xrightarrow{a.s.} X$. $\Rightarrow X_n \xrightarrow{P} X \Rightarrow X_n \xrightarrow{D} X$

Also,

$$E[\,|X_n - X|^2] < (1/n)^2 \longrightarrow 0$$

$\Rightarrow X_n \xrightarrow{m.s.} X$

6) Let $X_i = 1$: i^{th} game won
$X_i = 0$: i^{th} game lost

$$E[X_i] = 0.5(1) + 0.5(0) = 0.5$$
$$E[X_i^2] = 0.5(1^2) + 0.5(0^2) = 0.5 \implies Var(X_i) = 0.25$$

$$Y = \sum_{i=1}^{100} X_i \qquad S_{100} = \frac{Y - 50}{5}$$

(a) $P(Y \geq 55) = P(5S_{100} + 50 \geq 55) = P(S_{100} \geq 1)$

$$= 1 - P(S_{100} \leq 1)$$
$$\overset{CLT}{\approx} 1 - (1/2 + erf(1))$$
$$= 1/2 - erf(1) \approx 0.16$$

(b) Let N: number of matches

$$Y = \sum_{i=1}^{N} X_i \qquad S_N = \frac{Y - N/2}{\sqrt{N}/2}$$

$$P(Y \geq 65) = P(\sqrt{N}/2\, S_N + N/2 \geq 65)$$
$$= P\left(S_N \geq \frac{65 - N/2}{\sqrt{N}/2}\right)$$
$$= 1 - P\left(S_N \leq \frac{65 - N/2}{\sqrt{N}/2}\right)$$
$$= 1 - \left(1/2 - erf\left(\frac{N/2 - 65}{\sqrt{N}/2}\right)\right)$$

$$= 1/2 + \mathrm{erf}\left(\frac{N/2 - 65}{\sqrt{N}/2}\right) = 0.9$$

$$\Rightarrow \frac{N/2 - 65}{\sqrt{N}/2} = 1.30$$

$$\Rightarrow N - 130 = 1.30\sqrt{N}$$

$$\Rightarrow N^2 - 260N + (130)^2 = 1.69N$$

$$\Rightarrow N^2 - 261.69N + (130)^2 = 0$$

$$\Rightarrow N = \frac{261.69 + \sqrt{(261.69)^2 - 4(130)^2}}{2}$$

(which is $N = 146$, but I used my calculator!)