ECE 603 - Probability and Random Processes, Fall 2016

Midterm Exam #2

November 16, 7:00-9:00pm

Integrated Learning Center (ILC), Room S240

Overview

- The exam consists of five problems for 130 points. The points for each part of each problem are given in brackets - you should spend your **two hours** accordingly.
- The exam is closed book, but you are allowed **two page-sides** of notes. Calculators are not allowed. I will provide all necessary blank paper.

Testmanship

- **Full credit will be given only to fully justified answers.**
- Giving the steps along the way to the answer will not only earn full credit but also maximize the partial credit should you stumble or get stuck. If you get stuck, attempt to neatly define your approach to the problem and why you are stuck.
- If part of a problem depends on a previous part that you are unable to solve, explain the method for doing the current part, and, if possible, give the answer in terms of the quantities of the previous part that you are unable to obtain.
- Start each problem on a new page. Not only will this facilitate grading but also make it easier for you to jump back and forth between problems.
- If you get to the end of the problem and realize that your answer must be wrong, be sure to write "this must be wrong because . . . " so that I will know you recognized such a fact.
- Academic dishonesty will be dealt with harshly - the *minimum penalty* will be an "F" for the course.

1. The random variables X and Y have joint probability density function $f_{X,Y}(x,y)$ given by:

$$f_{X,Y}(x,y) = \begin{cases} c\,x, & 0 \le x \le 1,\ 0 \le |y| \le 1-x \\ 0, & \text{else} \end{cases}$$

[5] (a) Find c.

[10] (b) Find $f_X(x)$ and $f_Y(y)$, the marginal probability density functions of X and Y, respectively.

[8] (c) Find $f_{X|Y}(x|y)$, the conditional probability density function of X given Y. For your limits (which you should not forget!), put y between constant bounds and then give the limits for x in terms of y.

[5] (d) Are X and Y independent?

[10] (e) Consider $f_{X,Y}(x,y)$ as given above. Suppose that you want to obtain a large value of X (think of it being money). Suppose that you can choose $Y = 0.25$ or $Y = 0.75$, and then generate X given that setting of Y. Which Y do you choose? (**There might be multiple good answers. Just be sure to justify yours.**)

[7] (f) Consider $f_{X,Y}(x,y)$ as given above. Suppose I define the random variable $Z = Y - X$. Write an integral expression for $F_Z(z) = P(Z \le z)$, the cumulative distribution function of Z. You do not have to evaluate the integral, but be sure that all integral limits are precisely given.

2. Let X and Y be jointly Gaussian with means $E[X] = 1$, $E[Y] = 2$; variances $\sigma_X^2 = 9$, $\sigma_Y^2 = 16$; and correlation coefficient $\rho_{X,Y} = 0.4$.

[5] (a) Are X and Y independent?

[10] (b) For X and Y as defined above, define a sequence of random variables by $Z_n = \frac{2X}{n} + \frac{Y}{n}$. Find $P(Z_1 > 3)$.

[10] (c) For X and Y as defined above, define a sequence of random variables by $Z_n = \frac{2X}{n} + \frac{Y}{n}$. Does Z_n converge in mean square, probability, and/or distribution? If so, to what and in what ways? (You can use one form of convergence to imply another, if you like.)

3. [10] I have a random variable ω that is Laplacian; that is,

$$f_\omega(x) = \frac{1}{2}e^{-|x|}, \qquad -\infty < x < \infty$$

I have a series (in n) of circuits that limit the amplitude of their output to a given level, with increasing levels as n increases, and I want to consider whether the output of my circuits converges in some sense to the actual random variable (or anything else). Let:

$$Z_n(\omega) = \begin{cases} n, & \omega \ge n \\ \omega, & -n \le \omega < n \\ -n, & \omega < -n \end{cases}$$

Does Z_n converge, and, if so, to what and in what ways? Consider almost sure convergence, mean square convergence, convergence in probability, and convergence in distribution. You can do them in any order that you want, and you can use one type of convergence to imply another when appropriate.

4. You *may* find the following facts useful:

   - A binomial random variable Z_1 with probability mass function $P(Z_1 = k) = \binom{n}{k} p^k (1-p)^{(n-k)}$ has $E[Z_1] = np$.
   - A geometric random variable Z_2 with probability mass function $P(Z_2 = k) = (1-p)^{k-1}p$ has $E[Z_2] = 1/p$.

   Consider the following experiment: First, I draw a discrete random variable $X \in (\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1)$ according to the probability mass function:

$$P(X = p) = \begin{cases} \frac{1}{4}, & p = \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1 \\ 0, & \text{otherwise} \end{cases}$$

   Then, I take a coin with probability of heads X and flip it until I get a "heads"; I record the number of flips as Y. This forms an ordered pair of random variables (X, Y).

   [5] (a) Given $X = p$, find the probability mass function of Y; that is, find $P(Y = k|X = p)$.

   [10] (b) Suppose $Y = 2$. Evaluate (give numbers for) the conditional probability mass function $P(X = p|Y = 2)$, for $p = \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1$. What is the most probable value of X given $Y = 2$?

   [10] (c) Find $E[X^2 Y]$. (Your answer should be a number.)

5. *Convergence Problems:*

   [8] (a) Let the probability space $(\Omega, \mathcal{A}, P)$ be given by $\Omega = [0, 1]$, $\mathcal{A} = \mathcal{B}$ (restricted to [0,1], of course), and $P((a, b)) = b - a$. Let:

$$X_n(\omega) = \begin{cases} \frac{(-1)^n}{n} + \frac{\omega}{n} + 1, & \omega \in \text{irrationals} \\ (-1)^n + 5, & \{\omega \in \text{rationals}\} \cap \{0 \le \omega < \frac{1}{2}\} \\ (-1)^n + 10, & \{\omega \in \text{rationals}\} \cap \{\frac{1}{2} \le \omega \le 1\} \end{cases}$$

   Does the sequence X_n converge, and, if so, to what and in what ways? Consider almost sure convergence, mean square convergence, convergence in probability, and convergence in distribution. You can do them in any order that you want, and you can use one type of convergence to imply another when appropriate.

   [8] (b) Let the probability space $(\Omega, \mathcal{A}, P)$ be given by $\Omega = [0, 1]$, $\mathcal{A} = \mathcal{B}$ (restricted to [0,1], of course), and $P((a, b)) = b - a$. Let $X_n(\omega) = \omega n$. Does the sequence X_n converge, and, if so, to what and in what ways? Consider almost sure convergence, mean square convergence, convergence in probability, and convergence in distribution. You can do them in any order that you want, and you can use one type of convergence to imply another when appropriate.

   [9] (c) Let the probability space $(\Omega, \mathcal{A}, P)$ be given by $\Omega = [0, 1]$, $\mathcal{A} = \mathcal{B}$ (restricted to [0,1], of course), and $P((a, b)) = b - a$. We generate ω from this space and note that any ω has a unique binary expansion $\omega = 0.b_1 b_2 b_3 b_4 \ldots$, where $b_i \in \{0, 1\}$. Now, let $X_n(\omega) = b_n$ (i.e. the n^{th} digit of the binary expansion of ω). Does the sequence X_n converge, and, if so, to what and in what ways? Consider almost sure convergence, mean square convergence, convergence in probability, and convergence in distribution. You can do them in any order that you want, and you can use one type of convergence to imply another when appropriate.