ECE 603 - Probability and Random Processes, Fall 2010

Midterm Exam #2

November 22nd, 6:00-8:00pm, Goessman 20

Overview

- The exam consists of four problems for 130 points. The points for each part of each problem are given in brackets - you should spend your **two hours** accordingly.
- The exam is closed book, but you are allowed **two page-sides** of notes. Calculators are not allowed. I will provide all necessary blank paper.

Testmanship

- **Full credit will be given only to fully justified answers.**
- Giving the steps along the way to the answer will not only earn full credit but also maximize the partial credit should you stumble or get stuck. If you get stuck, attempt to neatly define your approach to the problem and why you are stuck.
- If part of a problem depends on a previous part that you are unable to solve, explain the method for doing the current part, and, if possible, give the answer in terms of the quantities of the previous part that you are unable to obtain.
- Start each problem on a new page. Not only will this facilitate grading but also make it easier for you to jump back and forth between problems.
- If you get to the end of the problem and realize that your answer must be wrong, be sure to write "this must be wrong because ..." so that I will know you recognized such a fact.
- Academic dishonesty will be dealt with harshly - the *minimum penalty* will be an "F" for the course.

You may find the following fact useful as you solve this exam:

Fact: Given a collection of jointly Gaussian random variables, every linear combination is Gaussian. Conversely, given that every linear combination of a collection of random variables is Gaussian, that collection of random variables is jointly Gaussian.

1. The random variables X and Y have joint probability density function:

$$f_{X,Y}(x,y) = \begin{cases} c(x+y), & 0 \le y \le 2, \frac{y}{2} \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

   [10] (a) Find c.

   [5] (b) Find $f_X(x)$, the marginal probability density function of X.

   [5] (c) Find $f_{Y|X}(y|x)$, the conditional probability density function of Y given X. **For this, write your limits for x between constants and your limits on y as (potentially) functions of x.**

   [5] (d) Suppose that you are planning to estimate Y from an observation X. Find the x such that when $X = x$, you are "best" able to estimate Y from $X = x$. In other words, if you are trying to estimate Y and are hoping for an $X = x$ that makes it easy, what x would you choose?

2. *More multiple random variables X and Y* (**Note: Some of these answers are short**):

   [7] (a) Let X be a Gaussian random variable with mean 0 and variance 1. Given $X = x$, let Y be uniformly distributed between 0 and $|x|$. Find $P(Y > X)$.

   [5] (b) Let X be a random variable uniformly distributed between 0 and 1. Given $X = x$, let Y be Gaussian with mean x and variance 1. Find $P(Y > X)$.

   [8] (c) Let X be a random variable uniformly distributed between 0 and 1. Let let Y be Gaussian with mean 0 and variance 1. Suppose X and Y are independent. Find $P(Y > X)$ (you can leave an integral expression here).

   [10] (d) Let the probability space $(\Omega, \mathcal{A}, P)$ be given by $\Omega = [0,2]$, $\mathcal{A} = \mathcal{B}$ (restricted to $[0,2]$, of course), and $P((a,b)) = c\,(b^2 - a^2)$.

   - Find c.
   - Suppose $X(\omega) = \omega, Y(\omega) = \omega^2$. Find $P(Y > X)$.

3. Suppose that I am interested in characterizing two random variables X and Y (both individually and their relation to each other). I know in advance that the variances of the two random variables are equal ($\sigma_X^2 = \sigma_Y^2 = \sigma^2$), but I do not know σ^2. I know nothing about either mean, including whether they are equal or not. Unfortunately, I am not able to observe the random variables individually, but only $X + Y$ and $X - Y$. Making a bunch of measurements, I come up with the following:

   - $E[X + Y] = 1$.
   - $E[X - Y] = -3$.
   - $E[(X + Y)^2] = 15$.
   - $E[(X - Y)^2] = 11$.

   [5] (a) Find $E[X]$ and $E[Y]$.

   [10] (b) Find σ^2 and $\rho_{X,Y}$, the correlation coefficient of X and Y.

   [15] (c) You have some extra time on your hands, so you define $Z = X + Y$, $W = X - Y$ and estimate $f_Z(z)$ and $f_W(w)$, the probability density functions Z and W, respectively. **Answer these three parts separately:**

   - Suppose that you note that Z is Gaussian and W is *not* Gaussian. Are X and Y jointly Gaussian? (**possible answers are yes, no, or maybe - a short justification is fine.**)
   - Suppose that you note that Z is Gaussian and W is Gaussian. Are X and Y jointly Gaussian? (**possible answers are yes, no, or maybe - a short justification is fine.**)
   - Suppose that you note that Z and W are jointly Gaussian. Are X and Y jointly Gaussian? (**possible answers are yes, no, or maybe - be rigorous here with your justification.**)

4. (a) Let the probability space $(\Omega, \mathcal{A}, P)$ be given by $\Omega = [0, 1]$, $\mathcal{A} = \mathcal{B}$ (restricted to $[0, 1]$, of course), and $P((a, b)) = b - a$. Let a sequence of random variables be defined by $X_n(\omega) = (\omega + \frac{1}{n})^2$.

   - [8] Find $f_{X_2|X_1}(x_2|x_1)$, the conditional probability density function of X_2 given X_1.
   - [12] Determine whether the sequence $\{X_n\}$ converges, and, if so, to what and in what ways? Consider almost sure convergence, mean square convergence, convergence in probability, and convergence in distribution. You can do them in any order that you want, and you can use one type of convergence to imply another when appropriate.

   (b) Let the probability space $(\Omega, \mathcal{A}, P)$ be given by $\Omega = [-1/2, 1/2]$, $\mathcal{A} = \mathcal{B}$ (restricted to $[-1/2, 1/2]$, of course), and $P((a, b)) = b - a$. Let a sequence of random variables be defined by $X_n(\omega) = (-1)^n \omega$.

   - [5] Find $f_{X_1}(x)$, the probability density function of X_1
   - [10] Determine whether the sequence $\{X_n\}$ converges, and, if so, to what and in what ways? Consider almost sure convergence, mean square convergence, convergence in probability, and convergence in distribution. You can do them in any order that you want, and you can use one type of convergence to imply another when appropriate.

   [10] (c) Let X_n be integer random variables such that $P(X_n = 0) = 1 - 1/n$. $P(X_n = i) = 1/n^2, i = 1, 2 \ldots n$. Determine whether the sequence $\{X_n\}$ converges, and, if so, to what and in what ways? Consider mean square convergence, convergence in probability, and convergence in distribution. You can do them in any order that you want, and you can use one type of convergence to imply another when appropriate.