ECE 603 - Probability and Random Processes, Fall 2009

Midterm Exam #2

November 18th, 6:00-8:00pm, ELAB 303

Overview

- The exam consists of five problems for 125 points. The points for each part of each problem are given in brackets - you should spend your **two hours** accordingly.
- The exam is closed book, but you are allowed **two page-side** of notes. Calculators are not allowed. I will provide all necessary blank paper.

Testmanship

- **Full credit will be given only to fully justified answers.**
- Giving the steps along the way to the answer will not only earn full credit but also maximize the partial credit should you stumble or get stuck. If you get stuck, attempt to neatly define your approach to the problem and why you are stuck.
- If part of a problem depends on a previous part that you are unable to solve, explain the method for doing the current part, and, if possible, give the answer in terms of the quantities of the previous part that you are unable to obtain.
- Start each problem on a new page. Not only will this facilitate grading but also make it easier for you to jump back and forth between problems.
- If you get to the end of the problem and realize that your answer must be wrong, be sure to write "this must be wrong because . . . " so that I will know you recognized such a fact.
- Academic dishonesty will be dealt with harshly - the *minimum penalty* will be an "F" for the course.

1. Consider the probability space $((0,1), \mathcal{B}, P)$, where $\mathcal{B}$ is restricted to $(0,1)$, of course, and $P(\cdot)$ is defined by:

$$P((a,b)) = \begin{cases} c\,(b^2 - a^2), & 0 \le a < b \le \frac{1}{2} \\ c\,(\frac{1}{4} - a^2) + c\,(b - \frac{1}{2}), & 0 \le a \le \frac{1}{2} < b \le 1 \\ c\,(b - a), & \frac{1}{2} < a < b \le 1 \end{cases}$$

where c is a constant.

[5] (a) Find c.

[7] (b) Find $P(|\omega - \frac{1}{2}| > \frac{1}{4})$, the probability that the outcome ω is more than $\frac{1}{4}$ away from $\frac{1}{2}$.

Define the following random variables:

$$X(\omega) = \begin{cases} 0, & 0 \le \omega \le 1/2 \\ 1, & 1/2 < \omega \le 1 \end{cases}$$

$$Y(\omega) = \begin{cases} 0, & 0 \le \omega \le 1/4 \\ 1, & 1/4 < \omega \le 3/4 \\ 0, & 3/4 < \omega \le 1 \end{cases}$$

[5] (c) Find $f_X(x)$, the probability density function of X.

[8] (d) Find $f_{X,Y}(x,y)$, the joint probability density function of X and Y. (**Three important notes: (1) Your solution to (c) does not really help you with this part, except as a check; (2) If you have a hard time writing the expression for the joint pdf, giving a different probabilistic description of X and Y is fine; (3) Note that you can do (e) without doing this part.**)

[5] (e) Find $E[XY]$.

2. Consider an experiment where I draw a piece of fruit from a shopping bag. The bag contains an Apple, Banana, Lime, Pear, and Orange. Since I draw them equally likely, I define the probability space $(S, \mathcal{A}, P)$ as:

$$S = \{\text{Apple, Banana, Lime, Pear, Orange}\}, \quad \mathcal{A} = \mathcal{P}_S \text{ (power set)}, \quad P(A) = \frac{|A|}{5}, \; A \in \mathcal{A}$$

The fruit rots at different rates depending on the type. Using advanced science, I calculate how "edible" a fruit is on day n to arrive at the sequence of random variables $\{X_n\}$ defined by:

$$X_n(\omega) = \begin{cases} 1/n, & \omega = \text{Apple} \\ 2/n, & \omega = \text{Banana} \\ 3/n, & \omega = \text{Lime} \\ 4/n, & \omega = \text{Pear} \\ 5/n, & \omega = \text{Orange} \end{cases}$$

[7] (a) Find $f_{X_1}(x)$ the probability density function of X_1.

[8] (b) Does $\{X_n\}$ converge? If so, to what and in what ways?

3. Let X and Y be random variables with joint probability density function:

$$f_{X,Y}(x,y) = \begin{cases} cx^2y, & -1 \le x \le 1, \ \ 0 \le y \le x^2 \\ 0, & \text{else} \end{cases}$$

[8] (a) Find c.

[7] (b) Find $P(X > \frac{1}{2})$.

[10] (c) Find $f_{X|Y}(x|y)$, the conditional probability density function of X given Y. For your limits, put constant bounds on y and then express x as a function of y.

[10] (d) Use your answer from (c) to find $f_{X|Y}(x|0.5)$, the conditional probability density function of X given $Y = 0.5$. Sketch $f_{X|Y}(x|0.5)$, just enough to indicate what values of X are most likely given $Y = 0.5$, and indicate what these values are.

4. Consider the sequence of random variables $\{X_n\}$, where $X_n \sim N((1 - 1/n), 1/n)$; that is, X_n is Gaussian with mean $(1 - 1/n)$ and variance $1/n$.

[5] (a) First consider just X_4. Find $P(X_4 > \frac{3}{2})$ in terms of the $\text{erf}(\cdot)$ function given in class:

$$\text{erf}(x) = \int_0^x \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$$

[10] (b) Repeat part (a), but now give $P(X_4 > \frac{3}{2})$ in terms of the "$\text{erf}(\cdot)$" function given on Wikipedia (which I spell with a capital "E", just to distinguish it from above):

$$\text{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

[10] (c) Finally, determine whether the sequence $\{X_n\}$ converges, and, if so, to what and in what ways? Consider mean square convergence, convergence in probability, and convergence in distribution. You can do them in any order that you want, and you can use one type of convergence to imply another when appropriate.

5. Expectations:

[10] (a) Suppose the random variable X has mean 3 and variance 9. Find constants a and b such that $W = aX + b$ has mean 0 and variance 1. (These would be called "normalization constants").

[10] (b) Let X and Y be jointly Gaussian random variables. Suppose you know that $E[X] = 0$ and $E[Y] = 0$. By doing measurements, you find that $E[(X + Y)^2] = 6$, and that $E[XY] = \frac{3}{2}$. Find possible values for the ordered pair (σ_X^2, σ_Y^2), the variances of X and Y, respectively. (*Note: Full credit goes to the smallest possible set of ordered pairs given the constraints above. Think very carefully about all of the constraints on the variances of X and Y.*)