ECE 603 - Probability and Random Processes, Fall 2008

Midterm Exam #2

November 13th, 6:00-8:00pm, Agricultural Engineering 119

Overview

- The exam consists of five problems for 120 points. The points for each part of each problem are given in brackets - you should spend your **two hours** accordingly.
- The exam is closed book, but you are allowed **two page-side** of notes. Calculators are not allowed. I will provide all necessary blank paper.

Testmanship

- **Full credit will be given only to fully justified answers.**
- Giving the steps along the way to the answer will not only earn full credit but also maximize the partial credit should you stumble or get stuck. If you get stuck, attempt to neatly define your approach to the problem and why you are stuck.
- If part of a problem depends on a previous part that you are unable to solve, explain the method for doing the current part, and, if possible, give the answer in terms of the quantities of the previous part that you are unable to obtain.
- Start each problem on a new page. Not only will this facilitate grading but also make it easier for you to jump back and forth between problems.
- If you get to the end of the problem and realize that your answer must be wrong, be sure to write "this must be wrong because ..." so that I will know you recognized such a fact.
- Academic dishonesty will be dealt with harshly - the *minimum penalty* will be an "F" for the course.

1. The random variables X and Y have joint probability density function

$$f_{X,Y}(x,y) = \begin{cases} cx^2y^2, & 0 \le x \le 1, x \le |y| \le 1 \\ 0, & \text{otherwise} \end{cases}$$

   [8] (a) Find the value of c.

   [5] (b) Write an expression (no need to evaluate) for $P(Y < 1 - X)$.

   [10] (c) Find $f_{X|Y}(x|y)$, the conditional probability density function of X given $Y = y$. **For your limits (which you should not forget), use $-1 \le y \le 1$ and then bound x in terms of y.**

   [7] (d) Somebody tells you that $Y = 0.5$. Given this information, find the x_0 such that $P(|X - x_0| < 0.1)$ is maximized.

   [10] (e) Define the function $g(\cdot)$ as:

$$g(y) = \begin{cases} y^2, & y < 0 \\ 0, & \text{else} \end{cases}$$

   Let $U = g(Y)$. Find $f_U(u)$, the probability density function of U.

2. *Jointly Gaussian random variables:*

   [10] (a) Let X and Y be jointly Gaussian random variables. Let $E[X] = 0$, $E[Y] = 0$, $E[X^2] = 4$, $E[Y^2] = 4$, and $\rho_{X,Y} = \frac{1}{3}$. Define $Z = 2X - 3Y$. Find $P(Z > 3)$.

   [10] (b) Let X and Y be jointly Gaussian random variables. Suppose you know that $E[X] = 0$ and $E[Y] = 0$, and that X and Y are independent. By doing measurements, you find that $E[X^2Y^2] = 4$, and that $E[X^2 + Y^2] = 5$. Find possible values for the pair (σ_X^2, σ_Y^2), the variances of X and Y, respectively, **and (do not forget this part)** the correlation coefficient $\rho_{X,Y}$.

3. [10] Let the probability space $(\Omega, \mathcal{A}, P)$ be given by $\Omega = [0,1]$, $\mathcal{A} = \mathcal{B}$ (where the Borel field is restricted to $[0,1]$, of course) and, for any interval,

$$P((a,b)) = \begin{cases} \frac{b^2-a^2}{2}, & 0 \le a < b \le \frac{1}{4} \\ \frac{b^2-a^2}{2} & \frac{1}{4} \le a < b \le 1 \\ \frac{1}{2} + \frac{b^2-a^2}{2}, & a < \frac{1}{4} < b \le 1 \end{cases}$$

   Let $X(\omega) = \omega^2$. Find $E[X]$.

4. *Counterexamples:*

   [10] (a) Let the random variable Θ be uniformly distributed on $[0, 2\pi]$. Define $X = \cos(\Theta)$ and $Y = \sin(\Theta)$. Show that X and Y are uncorrelated but not independent. *Be rigorous here, particularly for the "not independent" part.*

   [10] (b) Give an example where $\{X_n\}$ converges in mean square to X but does not converge with probability one. (Be sure to give $(\Omega, \mathcal{A}, P)$, and the definition of the random variables $X_n(\omega)$.)

5. *Convergence:*

[10] (a) Let the probability space $(\Omega, \mathcal{A}, P)$ be given by $\Omega = [0,1]$, $\mathcal{A} = \mathcal{B}$ (restricted to $[0,1]$, of course), and $P((a,b)) = b - a$. For $\omega \in \Omega$, let $X_n(\omega) = 1 - \frac{\omega}{n}$. Does this sequence of random variables converge in some sense? If so, to what and in what senses (almost surely, in probability, in mean square, in distribution)? As always, be sure to justify your claims. If you claim it does not converge in any sense, just establish that it does not converge to a limit in one sense (almost surely, in probability, in mean square, in distribution) - your choice!

[10] (b) Let the probability space $(\Omega, \mathcal{A}, P)$ be given by $\Omega = [0,1]$, $\mathcal{A} = \mathcal{B}$ (restricted to $[0,1]$, of course), and $P((a,b)) = b - a$. For $\omega \in \Omega$, let

$$X_n(\omega) = \begin{cases} 1 + \frac{1}{n}, & \omega < \frac{1}{2} \\ 2 + \frac{1}{n}, & \omega \geq \frac{1}{2} \end{cases}$$

Does this sequence of random variables converge in some sense? If so, to what and in what senses (almost surely, in probability, in mean square, in distribution)? As always, be sure to justify your claims. If you claim it does not converge in any sense, just establish that it does not converge to a limit in one sense (almost surely, in probability, in mean square, in distribution) - your choice!

[10] (c) Let the probability space $(\Omega, \mathcal{A}, P)$ be given by $\Omega = [0,1]$, $\mathcal{A} = \mathcal{B}$ (where the Borel field is restricted to $[0,1]$, of course) and, for any interval,

$$P((a,b)) = \begin{cases} \frac{b^2 - a^2}{2}, & 0 \leq a < b \leq \frac{1}{2} \\ \frac{b^2 - a^2}{2} & \frac{1}{2} \leq a < b \leq 1 \\ \frac{1}{2} + \frac{b^2 - a^2}{2}, & a < \frac{1}{2} < b \leq 1 \end{cases}$$

A sequence of random variables is defined as follows. If ω is a rational number, $X_n(\omega) = (-1)^n$ for all n. If ω is an irrational number, $X_n(\omega) = \frac{\omega^2}{n}$ for all n. Does this sequence of random variables converge in some sense? If so, to what and in what senses (almost surely, in probability, in mean square, in distribution)? As always, be sure to justify your claims. If you claim it does not converge in any sense, just establish that it does not converge to a limit in one sense (almost surely, in probability, in mean square, in distribution) - your choice!