ECE 603 - Probability and Random Processes, Fall 2006

Midterm Exam #2

November 13th, 6:00-8:00pm, Marston 132

Overview

- The exam consists of six problems for 120 points. The points for each part of each problem are given in brackets - you should spend your **two hours** accordingly.
- The exam is closed book, but you are allowed **two page-side** of notes. Calculators are not allowed. I will provide all necessary blank paper.

Testmanship

- **Full credit will be given only to fully justified answers.**
- Giving the steps along the way to the answer will not only earn full credit but also maximize the partial credit should you stumble or get stuck. If you get stuck, attempt to neatly define your approach to the problem and why you are stuck.
- If part of a problem depends on a previous part that you are unable to solve, explain the method for doing the current part, and, if possible, give the answer in terms of the quantities of the previous part that you are unable to obtain.
- Start each problem on a new page. Not only will this facilitate grading but also make it easier for you to jump back and forth between problems.
- If you get to the end of the problem and realize that your answer must be wrong, be sure to write "this must be wrong because ..." so that I will know you recognized such a fact.
- Academic dishonesty will be dealt with harshly - the *minimum penalty* will be an "F" for the course.

1. You are on your first job and are asked to characterize a random variable X. By making a large number of observations, you form a normalized histogram (it integrates to 1) as shown below. Using a computer program, you fit the following function to your histogram (i.e. this function is your estimate of the probability density function of X):

$$f_X(x) = \frac{e^{-\frac{x^2}{6}+x-\frac{3}{2}}}{\sqrt{6\pi}}$$

[10] (a) Use the estimated probability density function $f_X(x)$ to find $E[X]$, $E[X^2]$, and Var$[X]$ (i.e. the expected value of X, the mean squared value of X, and the variance of X).

[10] (b) Let the random variable Y be defined by:

$$Y = \begin{cases} 3X, & X \geq 0 \\ 0, & X < 0 \end{cases}$$

Find $F_Y(y)$, the cumulative density function (CDF) of Y.

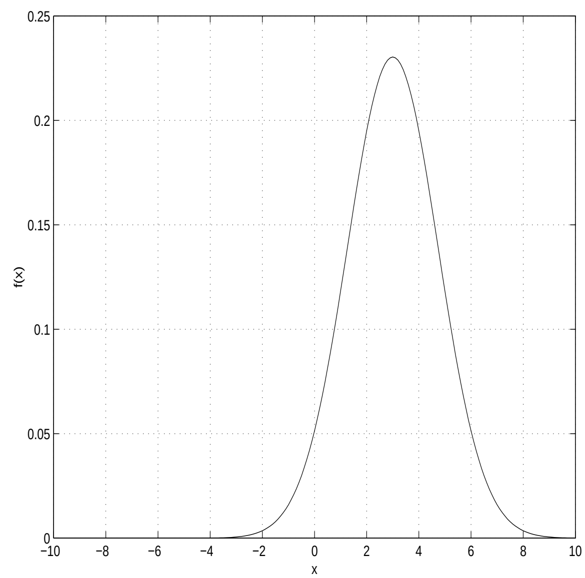


Figure 1: Normalized histogram of X for Problem 1.

2. Suppose that the random variable X is uniform on $[0, 1]$. Given $X = x$, Y is a uniform random variable on $[x, 1]$.

[12] (a) Find $f_Y(y)$, the marginal probability density function of Y.

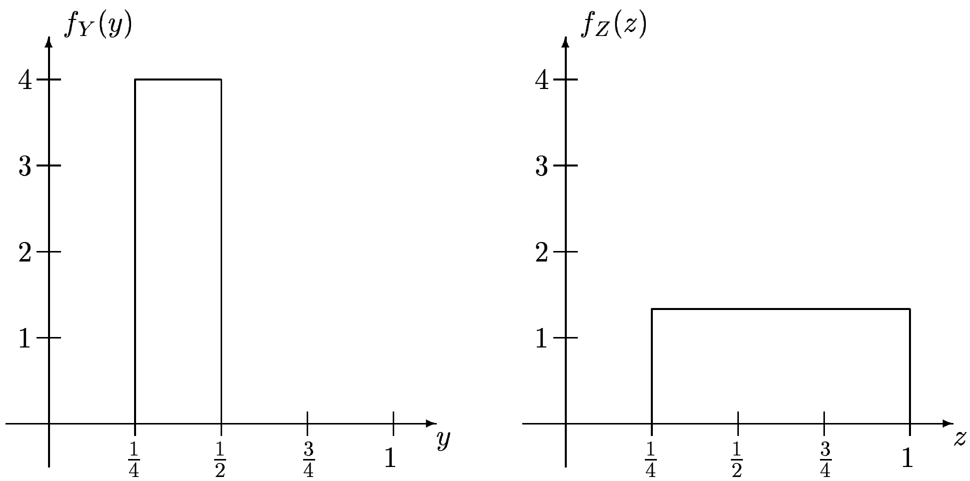
[8] (b) Find $P(Y > 1 - X)$, the probability that $Y > 1 - X$.

3. A coin is flipped three times. Let the random variable X be the number of **heads in the first two flips of the coin**, and let the random variable Y be the number of **tails in all three coin flips**.

   [8] (a) Find $P(X = x, Y = y)$ for all x and y. *The easiest way to express your answer is probably a table in x and y.*

   [7] (b) Find $\text{cov}(X, Y)$, the covariance of X and Y.

4. The money (in thousands of dollars) made from investing in stocks "Ystock" and "Zstock" are modeled as the random variables Y and Z, respectively. Assume Y and Z are **independent** with respective probability density functions $f_Y(y)$ and $f_Z(z)$ as shown below:



   [5] (a) You want to make as much money as possible, of course. Which stock would you buy?

   [10] (b) Suppose you decide to buy "Ystock" and your friend decides to buy "Zstock". What is the probability that you make more money? (In other words, find $P(Y > Z)$).

5. Convergence of random sequences:

[10] (a) Let the probability space $(S, \mathcal{F}, P)$ be given by $S = [0,1]$, $\mathcal{F} = \mathcal{B}$ (restricted to $[0,1]$, of course), and $P((a,b)) = b - a$. For $\omega \in S$, let $X_n(\omega) = \frac{1}{n}(1-\omega)$. Does this sequence of random variables converge? If so, to what and in what sense (almost surely, in probability, in mean square, in distribution)? If not, just establish that it does not converge to a limit in one sense (almost surely, in probability, in mean square, in distribution) - your choice!

[10] (b) Let the probability space $(S, \mathcal{F}, P)$ be given by $S = [0,2]$, $\mathcal{F} = \mathcal{B}$ (restricted to $[0,2]$, of course), and $P((a,b)) = (b-a)/2$. For $\omega \in S$, let $X_n(\omega) = \omega^n$. Does this sequence of random variables converge? If so, to what and in what sense (almost surely, in probability, in mean square, in distribution)? If not, just establish that it does not converge to a limit in one sense (almost surely, in probability, in mean square, in distribution) - your choice!

[10] (c) Let the probability space $(S, \mathcal{F}, P)$ be given by $S = [0,1]$, $\mathcal{F} = \mathcal{B}$ (restricted to $[0,1]$, of course), and $P((a,b)) = (b-a)$. For $\omega \in S$, let

$$X_n(\omega) = \begin{cases} 0, & 0 \leq \omega < \frac{1}{n} \\ \frac{1}{n}, & \frac{1}{n} \leq \omega < \frac{2}{n} \\ \frac{2}{n}, & \frac{2}{n} \leq \omega < \frac{3}{n} \\ \dots & \\ \frac{k}{n}, & \frac{k}{n} \leq \omega < \frac{k+1}{n} \\ \dots & \\ \frac{n-1}{n}, & \frac{n-1}{n} \leq \omega \leq 1 \end{cases}$$

Does this sequence of random variables converge? If so, to what and in what sense (almost surely, in probability, in mean square, in distribution)? If not, just establish that it does not converge to a limit in one sense (almost surely, in probability, in mean square, in distribution) - your choice!

6. Your team wins each match with probability 0.50 and loses each match with probability 0.50 (there are no ties). Assume that each match is **independent** of all other matches. Answer the following questions using the Central Limit Theorem.

[10] (a) Assuming that the season lasts 100 matches, find the probability that your team **wins** more than 55 matches.

[10] (b) Suppose that you wanted it such that your team wins at least 65 matches with probability 0.90. What is the minimum number of matches in a season for this to be true? *Hint: Feel free to use the Central Limit Theorem loosely as on the homework to solve this part.*