ECE 603 - Probability and Random Processes, Fall 2002

Midterm Exam #2

November 20th, 6:00-8:00pm, Marston 132

Overview

- The exam consists of six problems for 120 points. The points for each part of each problem are given in brackets - you should spend your **two hours** accordingly.
- The exam is closed book, but you are allowed **two page-sides** of notes. Calculators are not allowed. I will provide all necessary blank paper.

Testmanship

- **Full credit will be given only to fully justified answers.**
- Giving the steps along the way to the answer will not only earn full credit but also maximize the partial credit should you stumble or get stuck. If you get stuck, attempt to neatly define your approach to the problem and why you are stuck.
- If part of a problem depends on a previous part that you are unable to solve, explain the method for doing the current part, and, if possible, give the answer in terms of the quantities of the previous part that you are unable to obtain.
- Start each problem on a new page. Not only will this facilitate grading but also make it easier for you to jump back and forth between problems.
- If you get to the end of the problem and realize that your answer must be wrong, be sure to write "this must be wrong because . . . " so that I will know you recognized such a fact.
- Academic dishonesty will be dealt with harshly - the *minimum penalty* will be an "F" for the course.

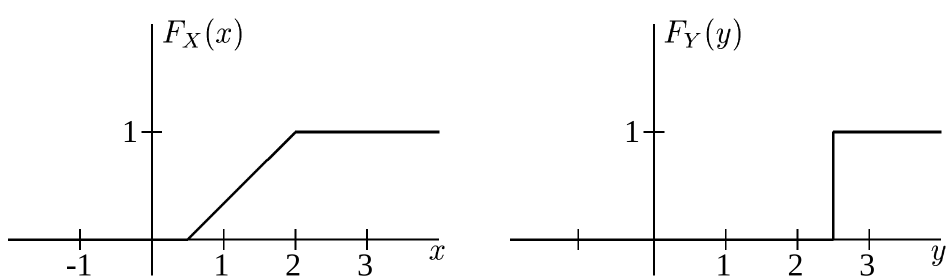
1. [10] A point is chosen at random within a circle of radius 3 (i.e. all points within the circle are equally likely). The outcome (or observation) for the experiment is the distance of the point from the center of the circle. Define a non-trivial probability space for this experiment; that is, find $(\Omega, \mathcal{A}, P)$, where Ω is the observation space, $\mathcal{A}$ is a set of subsets of Ω to which probabilities are assigned, and P is a probability mapping from $\mathcal{A}$ to $[0, 1]$.

2. A fair six-sided die is rolled **once**. Let N be the number of spots appearing on the top of the die. A coin is then flipped N times, and the number of heads (call it M) is recorded.

   [8] (a) Find the probability that $M = 5$.

   [7] (b) Given that $M = 5$, find the probability that $N = 6$.

3. Suppose that your goal is to maximize the profit of your business. If you decide to travel to Xville, the profit (in dollars) for your business is a random variable X with cumulative distribution function $F_X(x)$ as given below. If you decide to travel to Yville, the profit (in dollars) for the trip is a random variable Y with cumulative distribution function $F_Y(y)$ as given below. The random variables X and Y are independent.



   [8] (a) To which city (Xville or Yville) do you travel to maximize the profit for your company. **Be sure to justify your answer.**

   [7] (b) Suppose you have some extra time so you travel to both Xville and Yville. Let the random variable Z be your total profit from the two trips. Sketch the cumulative distribution function $F_Z(z) = P(Z \leq z)$ for the random variable z.

4. [20] An experiment is defined by the probability space $(\Omega, \mathcal{A}, P)$, where $\Omega = [0, 1]$, $\mathcal{A}$ is the Borel σ-algebra restricted to $[0, 1]$, and $P(\cdot)$ is defined

by $P((a,b)) = b-a$. A sequence of random variables is defined as follows. If ω is a rational number, $X_n(\omega) = 1$ for all n. If ω is an irrational number, $X_n(\omega) = \frac{\omega^2}{n}$ for all n. Does this sequence of random variables converge? If so, in what ways and to what limiting random variable does it converge?

5. I flip a coin repeatedly. If the i^{th} flip is "heads", I let $X_i = 1$. If the i^{th} flip is "tails", I let $X_i = 0$. Let $Y[n] = \sum_{i=1}^{n} X_i$.

   [5] (a) Find the probability mass function of $Y[n]$.

   [5] (b) Find $m_Y[n]$, the mean function of $Y[n]$.

   [10] (c) Find $R_Y[m,n]$, the autocorrelation function of $Y[n]$.

   [10] (d) Does the sequence of random variables $Y[n]$ converge? If so, in what ways and to what limiting random variable does it converge? [Be sure to justify your answer].

6. Let $X(t)$ and $Y(t)$ be zero-mean, wide-sense stationary Gaussian random processes that are independent of one another. (*Recall that a Gaussian random process is one for which any collection of samples is jointly Gaussian*). Assume that the two processes have the same autocorrelation function $R_X(\tau) = R_Y(\tau) = 5\frac{\sin(3\pi\tau)}{3\pi\tau}$. (Note that $R_X(0) = 5$). Define the random process $Z(t)$ as:

$$Z(t) = X(t)\cos(2\pi 20t) + Y(t)\sin(2\pi 20t)$$

   [5] (a) Find $m_Z(t)$, the mean function of $Z(t)$.

   [8] (b) Find $R_Z(t_1, t_2)$, the autocorrelation function of $Z(t)$.

   [10] (c) Find the probability density function $f_{Z(t)}(x)$.

   [7] (d) Pick any two times t_1 and t_2, $t_1 \neq t_2$, that you want (your choice!) and give the joint probability density function $f_{Z(t_1),Z(t_2)}(x_1, x_2)$ of the random variables $Z(t_1)$ and $Z(t_2)$. (For example, you might choose $t_1 = 0$ and $t_2 = 5$; then, you just have to give $f_{Z(0),Z(5)}(x_1, x_2)$).