ECE 603 — Fall, 2010

1) (a) $A_1 \cap A_2$, $\overline{A_1 \cap A_2}$ a partition

$$P(A_3) = P(A_3 \mid A_1 \cap A_2)\, P(A_1 \cap A_2) + P(A_3 \mid \overline{A_1 \cap A_2})\, P(\overline{A_1 \cap A_2})$$

A_1, A_2 indep

$$= P(A_3 \mid A_1 \cap A_2)\, P(A_1) P(A_2) + P(A_3 \mid \overline{A_1 \cap A_2})\,(1 - P(A_1)P(A_2))$$

$$= 0.4 \cdot 0.5 \cdot 0.5 + 1/3 \cdot 3/4$$

$$= 0.35$$

(b)

$$C = ((A_1 \cap A_2) \cup A_3) \cap (A_4 \cup A_5)$$

A_4, A_5 a partition

$$P(C) = P((A_1 \cap A_2) \cup A_3) \cdot \cancel{P(A_4 \cup A_5)}^{\,1}$$

$$= P(A_1 \cap A_2) + P(A_3) - P(A_3 \mid A_1 \cap A_2)\, P(A_1 \cap A_2)$$

$$= 0.5 \cdot 0.5 + 0.35 - 0.4 \cdot 0.5 \cdot 0.5$$

$$= 0.5$$

(c)

$$P(A_3 \mid C) = \frac{\cancel{P(C \mid A_3)}^{\,1}\, P(A_3)}{P(C)} = \frac{1 \cdot 0.35}{0.5} = 0.7$$

(d)

- Need to minimize $P(A_3 \mid A_1 \cap A_2)$

Can we make it zero? $0 + x \cdot 3/4 = 0.35$

$$x = 7/15 \Rightarrow P(A_3 \mid A_1 \cap A_2) = 0$$
$$P(A_3 \mid \overline{A_1 \cap A_2}) = 7/15$$

- Need to maximize $P(A_3 \mid A_1 \cap A_2)$

Can we make it one? $1 \cdot 0.5 \cdot 0.5 + x \cdot 3/4 = 0.35$

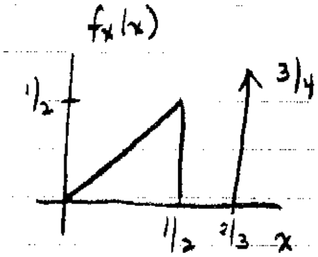
$$x = 2/15 \Rightarrow P(A_3 \mid A_1 \cap A_2) = 1$$
$$P(A_3 \mid \overline{A_1 \cap A_2}) = 2/15$$

2)

(a) $F_X(x) = P(X \le x)$

$$= \begin{cases} 0, & x < 0 \\ P((0,x)), & 0 \le x < 1/2 \\ P((0,1/2)), & 1/2 \le x < 2/3 \\ 1, & x \ge 2/3 \end{cases}$$

$$= \begin{cases} 0, & x < 0 \\ x^2, & 0 \le x < 1/2 \\ 1/4, & 1/2 \le x < 2/3 \\ 1, & x \ge 2/3 \end{cases}$$



$$f_X(x) = d/dx\, F_X(x) = \begin{cases} 2x, & 0 \le x < 1/2 \\ 0, & \text{else} \end{cases} + 3/4\, \delta(x - 2/3)$$

(b) • Denote it $(S, \mathcal{A}, P)$

$S = \{A, B, C, D\}$

$\mathcal{A} = \mathcal{P}_S$

$P(A) = (1/8)^2 - 0^2 = 1/64$

$P(B) = (1/4)^2 - (1/8)^2 = 3/64$

$P(C) = (1/2)^2 - (1/4)^2 = 3/16$

$P(D) = 1^2 - (1/2)^2 = 3/4$

and for any $A \in \mathcal{A}$, $P(A) = \sum_{x \in A} P(\{x\})$

• $P(R|F) = \dfrac{P(R \cap F)}{P(F)} = \dfrac{P(A)}{P(A) + P(B)} = \dfrac{1/64}{4/64} = 1/4$

3)

(a) $S = \{H, T\}^\infty$ which is 1-to-1 with $\{0,1\}^\infty$

$\Rightarrow$ uncountable

(b)

$S = \{H, T\}^\infty$

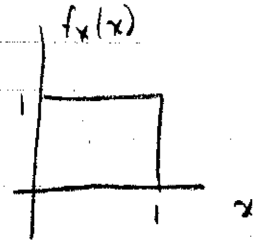
Now for each ω, use $H \to 1$, $T \to 0$, and write as:

$$0.b_1 b_2 b_3 \ldots \qquad b_i \in \{0,1\}$$

Now, use $\mathcal{A} = \mathbb{B}$, and, since all outcomes are equally likely,

$$P((a,b)) = b - a.$$

4)



(a) Each instance just increased by 2: shift pdf $\Rightarrow f_Y(y) = \begin{cases} 1, & 2 \le y \le 3 \\ 0, & \text{else} \end{cases}$

(b) Each instance just negated: flip pdf $\Rightarrow f_Y(y) = \begin{cases} 1, & -1 \le y \le 0 \\ 0, & \text{else} \end{cases}$

(c) All x's get mapped into 5 $\Rightarrow f_Y(y) = \delta(x-5)$

(d) $E[Z] = 1 \cdot 1/3 + 2 \cdot 2/3 = 5/3$

$E[Z^2] = 1^2 \cdot 1/3 + 2^2 \cdot 2/3 = 3$

(e) $Z=1 \Rightarrow Y=18$

$Z=2 \Rightarrow Y=36$

$f_Y(y) = 1/3 \cdot \delta(y-18) + 2/3 \cdot \delta(y-36)$

(f) Need more than 5 2's.

$$\sum_{l=6}^{10} \binom{10}{l} (2/3)^l (1/3)^{10-l}$$

5)

$$f_X(x) = c\,e^{x - x^2/6}$$

(a) Looks Gaussian with mean 3 - try that

$$f_X(x) = c\,e^{-1/6\,(x^2 - 6x)}$$

$$= c\,e^{-1/6\,(x^2-6x+9)}\,e^{3/2}$$

$$= c\,e^{3/2}\,e^{-(x-3)^2/2\cdot 3} \qquad X \sim N(3,3)$$

$$\Rightarrow c\,e^{3/2} = 1/\sqrt{2\pi 3} \Rightarrow c = 1/\sqrt{6\pi} \cdot e^{3/2}$$

$$\Rightarrow f_X(x) = 1/\sqrt{2\pi \cdot 3}\; e^{-(x-3)^2/2\cdot 3}$$

(b) $E[X^2] = Var(X) + (E[X])^2 = 3 + 3^2 = 12$

(c) $P(X>6) = 1 - P(X \le 6) = 1 - (1/2 + erf(\sqrt{3})) = 1/2 - erf(\sqrt{3})$

$$\frac{x-\mu}{\sigma} = \frac{6-3}{\sqrt{3}} = \sqrt{3}$$

(d) $E[5X^2 + 4X + 2]$

$= 5\,E[X^2] + 4\,E[X] + 2$

$= 5 \cdot 12 + 4 \cdot 3 + 2 = 74$