Midterm 1 Solutions

Fall, 2015

1) Consider a fixed n. There are $\mathbb{Z}^n$ such polynomials and each has (at most) n irrational solutions. Call this set of solutions A_n. Then,

$$A = \bigcup_{n=1}^{\infty} A_n$$

is a countable union of countable sets. Hence A is countable

2)

(a) $P(Q_2) \overset{\text{Law of Total Probability}}{=} P(Q_2|Q_1)P(Q_1) + P(Q_2|\overline{Q}_1)P(\overline{Q}_1)$ — $Q_1, \overline{Q}_1$ a partition

$= 0.9 \cdot 0.9 + 0.3 \cdot 0.1 = 0.84$

(b) $P(Q_1|Q_2) \overset{\text{Bayes Rule}}{=} \dfrac{P(Q_2|Q_1)P(Q_1)}{P(Q_2)} = \dfrac{0.9 \cdot 0.9}{0.84} = \dfrac{0.81}{0.84}$

(c) $P(Q_1|Q_3) \overset{\text{independent}}{=} P(Q_1) = 0.9$

(d) • Need Q_1 and (Q_2 or Q_3)

$A = Q_1 \cap (Q_2 \cup Q_3)$

• $P(A) = ((Q_1 \cap Q_2) \cup (Q_1 \cap Q_3))$

$= P(Q_1 \cap Q_2) + P(Q_1 \cap Q_3) - P(\underbrace{(Q_1 \cap Q_2) \cap (Q_1 \cap Q_3)}_{Q_1 \cap Q_2 \cap Q_3})$

$= P(Q_2|Q_1)P(Q_1) + P(Q_1)P(Q_3) - P(Q_3)P(Q_2|Q_1)P(Q_1)$

$= 0.9 \cdot 0.9 + 0.9 \cdot 0.5 - 0.5 \cdot 0.9 \cdot 0.9$

$= 0.81 + 0.45 - 0.405$

$= 0.855$

3) (a)

Lots of ways to do this. Let A_i: no repeats after roll i.

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) \cdot P(A_2 \mid A_1) \cdot P(A_3 \mid A_1, A_2)$$

$$= 1 \cdot 2/3 \cdot 1/3 = 2/9$$

(b) Bernoulli with $p = 1/3$

$$\binom{20}{8} \left(1/3\right)^8 \left(2/3\right)^{12}$$

(c)

A: event I see at least one of each

R_0: no red

B_0: no blue

G_0: no green

$$P(A) = 1 - P(\bar{A})$$

$$= 1 - P(R_0 \cup B_0 \cup G_0)$$

$$= 1 - \left(P(R_0 \cup B_0) + P(G_0) - P((R_0 \cup B_0) \cap G_0)\right)$$

$$= 1 - \left(P(R_0) + P(B_0) - P(R_0 \cap B_0) + P(G_0) - P((R_0 \cap G_0) \cup (B_0 \cap G_0))\right)$$

disjoint

$$= 1 - \left(P(R_0) + P(B_0) + P(G_0) - P(R_0 \cap B_0) - P(R_0 \cap G_0) - P(B_0 \cap G_0)\right)$$

$$= 1 - \left(\left(2/3\right)^n + \left(2/3\right)^n + \left(2/3\right)^n - \left(1/3\right)^n - \left(1/3\right)^n - \left(1/3\right)^n\right)$$

$$= 1 - 3 \cdot \left(2/3\right)^n + 3 \cdot \left(1/3\right)^n$$

(e)

Think about each outcome separately.

$$P(D=0) = P(\{D_1=0\} \cup \{D_2=0\})$$

$$= P(D_1=0) + P(D_2=0) - \overbrace{P(D_1=0)P(D_2=0)}^{D_1, D_2 \text{ independent}}$$

$$= 1/2 + 1/10 - 1/2 \cdot 1/10$$

$$= 0.55$$

$$P(D=3) = P(\{D_1=3\} \cap \{D_2=3\})$$

$$= P(D_1=3) \cdot P(D_2=3)$$

$$= 1/10 \cdot 3/10 = 0.03$$

$$P(D=2) = P((\{D_1=2\} \cap \{D_2=2\}) \cup (\{D_1=2\} \cap \{D_2=3\}) \cup (\{D_1=3\} \cap D_2=2\}))$$

disjoint, independent

$$= 1/10 \cdot 3/10 + 1/10 \cdot 3/10 + 1/10 \cdot 3/10$$

$$= 0.09$$

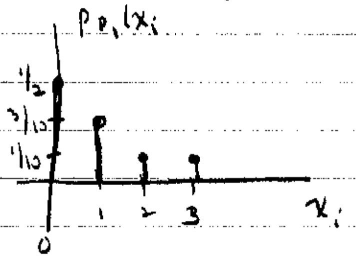
$$P(D=1) = 1 - P(D=0) - P(D=1) - P(D=2)$$

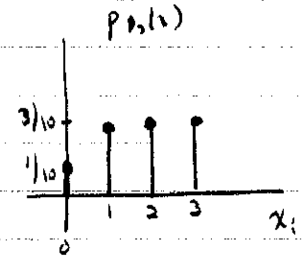
$$= 1 - 0.55 - 0.03 - 0.09$$

$$= 0.33$$

5)

Convert to pmfs





(a) Warehouse $\underline{1}$!

$$P(D_1 \le x) \ge P(D_2 \le x) \quad \text{for all } x$$

(b)

$$P(D \ge 2) = P(D \ge 2 \mid w_1)\,P(w_1) + P(D \ge 2 \mid w_2)\,P(w_2)$$

$$= 2/10 \cdot 1/2 + 6/10 \cdot 1/2$$

$$= 2/5$$

(c)

$$P(w_2 \mid D=2) = \frac{P(\{D=2\} \cap w_2)}{P(D=2)} = \frac{P(D=2 \mid w_2)\,P(w_2)}{P(D=2 \mid w_2)\,P(w_2) + P(D=2 \mid w_1)\,P(w_1)}$$

$$= \frac{3/10 \cdot 1/2}{3/10 \cdot 1/2 + 1/10 \cdot 1/2} = 3/4$$

(d) $P(\text{"success"} \mid w_1) = 1/2$

$P(\text{"success"} \mid w_2) = 1/10$

A: event of interest

$$\underbrace{\binom{10}{7}(1/2)^7(1/2)^3}_{P(A \mid w_1)} \cdot \underbrace{1/2}_{P(w_1)} + \underbrace{\binom{10}{7}(1/10)^7(9/10)^3}_{P(A \mid w_2)} \cdot \underbrace{1/2}_{P(w_2)}$$

(b)

$$F_w(x) = P(w \leq x)$$

$$\overset{x \geq 0}{=} P((0, x))$$

$$= \begin{cases} x - \frac{1}{2}x^2, & 0 \leq x < \frac{1}{2} \\ \frac{1}{2} + x - \frac{1}{2}x^2, & x \geq \frac{1}{2} \\ 0, & x < 0 \\ 1, & x > 1. \end{cases}$$

$$f_w(x) = \frac{d}{dx} F_w(x) = \begin{cases} 1 - x, & 0 \leq x < 1 \\ 0, & \text{else} \end{cases} + \frac{1}{2}\delta(x - \frac{1}{2})$$

$$E[w^2] = \int_0^1 x^2(1-x)\,dx + \int_0^1 x^2 \cdot \frac{1}{2} \cdot \delta(x - \frac{1}{2})\,dx$$

$$= \left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 + \left(\frac{1}{2}\right)^2 \cdot \frac{1}{2}$$

$$= \frac{1}{3} - \frac{1}{4} + \frac{1}{8} = \frac{8}{24} - \frac{6}{24} + \frac{3}{24} = \frac{5}{24}$$

4)

(a) We know $P(0,1) = 1$

$$\tfrac{1}{2} + c\cdot\left((1-0) - \tfrac{1}{2}(1-0)^2\right)$$

$$= \tfrac{1}{2} + \tfrac{1}{2}c$$

$$\Rightarrow c = 1$$

(b) ~~on next page~~

(c) $S_4 = \{A, B, C, D\}$

$$\mathcal{Y} = \mathcal{P}_{S_4} = \{\emptyset, \{A\}, \{B\}, \{C\}, \{D\}, \{A,B\}, \{A,C\}, \{A,D\}, \{B,C\}, \{B,D\}, \{C,D\}, \{A,B,C\}, \{A,B,D\}, \{A,C,D\}, \{B,C,D\}, S\}$$

$$\left.\begin{aligned}
P(\{A\}) &= P((0, 0.3)) = 0.3 - \tfrac{1}{2}\cdot 0.3^2 = 0.255 \\
P(\{B\}) &= P((0.3, 0.6)) = \tfrac{1}{2} + 0.3 - \tfrac{1}{2}\cdot\underbrace{(0.6^2 - 0.3^2)}_{0.27} = 0.665 \\
P(\{C\}) &= P((0.6, 0.9)) = 0.3 - \tfrac{1}{2}\underbrace{(0.9^2 - 0.6^2)}_{0.45} = 0.075 \\
P(\{D\}) &= P((0.9, 1)) = 0.1 - \tfrac{1}{2}\underbrace{(1 - 0.9^2)}_{0.19} = 0.005
\end{aligned}\right\} \text{sum to } 1!$$

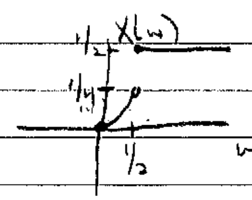
$$P(E) = \sum_{x_i \in E}{}' P(\{x_i\}) \qquad x_i \in \{A, B, C, D\}$$

(d)

$$F_X(x) = P(X \le x) = \overset{x \le 1/4}{P}(w^2 \le x)$$

$$= P(w \le \sqrt{x})$$

$$= P((0, \sqrt{x}))$$



$$F_X(x) = \begin{cases} \sqrt{x} - \tfrac{1}{2}x, & 0 \le x < \tfrac{1}{4} \\ 7/8, & \tfrac{1}{4} \le x < \tfrac{1}{2} \\ 1, & x \ge \tfrac{1}{2} \end{cases}$$

$$\Rightarrow f_X(x) = \begin{cases} \tfrac{1}{2\sqrt{x}} - \tfrac{1}{2}, & 0 \le x < \tfrac{1}{4} \\ 0, & \end{cases} + \tfrac{1}{2}\delta(x - \tfrac{1}{4}) + \tfrac{1}{8}\delta(x - \tfrac{1}{2})$$

(e) $w \le 1 \Rightarrow Y \le 5 \Rightarrow P(Y > 6) = 0.$