ECE 603 - Probability and Random Processes, Fall 2009

Midterm Exam #1

October 22nd, 6:00-8:00pm, ELAB 303

Overview

- The exam consists of five problems for 120 points. The points for each part of each problem are given in brackets - you should spend your **two hours** accordingly.
- The exam is closed book, but you are allowed **one page-side** of notes. Calculators are not allowed. I will provide all necessary blank paper.

Testmanship

- **Full credit will be given only to fully justified answers.**
- Giving the steps along the way to the answer will not only earn full credit but also maximize the partial credit should you stumble or get stuck. If you get stuck, attempt to neatly define your approach to the problem and why you are stuck.
- If part of a problem depends on a previous part that you are unable to solve, explain the method for doing the current part, and, if possible, give the answer in terms of the quantities of the previous part that you are unable to obtain.
- Start each problem on a new page. Not only will this facilitate grading but also make it easier for you to jump back and forth between problems.
- If you get to the end of the problem and realize that your answer must be wrong, be sure to write "this must be wrong because ..." so that I will know you recognized such a fact.
- Academic dishonesty will be dealt with harshly - the *minimum penalty* will be an "F" for the course.

Hint: You may find the following fact useful as you solve this exam:

$$P\left(\bigcap_{n=1}^{\infty}(x-\frac{1}{n},x+\frac{1}{n})\right)=\lim_{n\to\infty}P((x-\frac{1}{n},x+\frac{1}{n}))$$

1. Consider the probability space $((0,2),\mathcal{B},P)$, where $\mathcal{B}$ is restricted to $(0,2)$, of course, and $P(\cdot)$ is defined by:

$$P((a,b))=\begin{cases} c\,(b^2-a^2), & 0\le a<b\le\frac{2}{3} \\ c\,(b^2-a^2), & \frac{2}{3}\le a<b\le 2 \\ \frac{3}{4}+c\,(b^2-a^2), & a<\frac{2}{3}<b\le 2 \end{cases}$$

where c is a constant. Let X be the outcome of the experiment.

[7] (a) What is $P(\frac{1}{2}<X<1)$? *Your answer should be a number.*

[8] (b) What is $P(\frac{2}{3}\le X<2)$? *Be sure to derive everything from first principles!*

[5] (c) Since $X\in\mathcal{R}$, it can be treated as a random variable. Find and *roughly* sketch the cumulative distribution function (CDF) $F_X(x)$ and probability density function $f_X(x)$ of X.

[10] (d) Let the random variable Y be defined as $Y=X^2-2$. Find the probability space $(S,\mathcal{A},P_Y)$ for Y.

2. You play a carnival game that consists of throwing three darts at a target. Let D_i be the event that dart i hits the target. Suppose you have the following information:

- D_1 is independent of D_2 and D_3
- $P(D_1)=0.8$
- $P(D_2)=0.8$
- $P(D_3|D_2)=0.9$.
- $P(D_3|\overline{D}_2)=0.5$.

[5] (a) Find $P(D_3)$, the probability that the third dart hits the target.

[5] (b) Given that the third dart hits the target, what is the probability that the second dart hit the target.

[5] (c) Find the probability that all three darts hit the target.

[5] (d) Find the probability that exactly two darts hit the target.

3. An exponential random variable with parameter λ has probability density function:

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{else} \end{cases}$$

Suppose that I have three lightbulbs of varying qualities:

- The first dies out after L_1 hours, where L_1 is an exponential random variable with $\lambda = 1$.
- The second dies out after L_2 hours, where L_2 is an exponential random variable with $\lambda = 2$.
- The third dies out after L_3 hours, where L_3 is an exponential random variable with $\lambda = 3$.

Definition: A random variable X is said to be *memoryless* if $P(\{X \geq (x+l)\}|\{X \geq x\}) = P(\{X \geq l\})$ for all x and l.

[5] (a) Suppose that I use the first lightbulb (which lasts a random time L_1) in my lamp, and it has lasted two hours. Given such, what is the chance it lasts more than four hours?

[5] (b) Is L_1 memoryless?

[7] (c) I pick a lightbulb at random (i.e. each of the three is equally likely) and put it into my lamp. Denote the time that the lamp is lit as L. What is the probability that the lamp is lit for more than 2 hours?

[8] (d) I pick a lightbulb at random (i.e. each of the three is equally likely) and put it into my lamp. Denote the time that the lamp is lit as L. The lamp is lit for two hours. Given such, what is the probability that it is lit for more than four hours?

[5] (e) Is L memoryless? Explain any difference from (b).

4. The probability density function of a random variable X is given by $f_X(x)$, where:

$$f_X(x) = \begin{cases} c\,|x| & -1 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

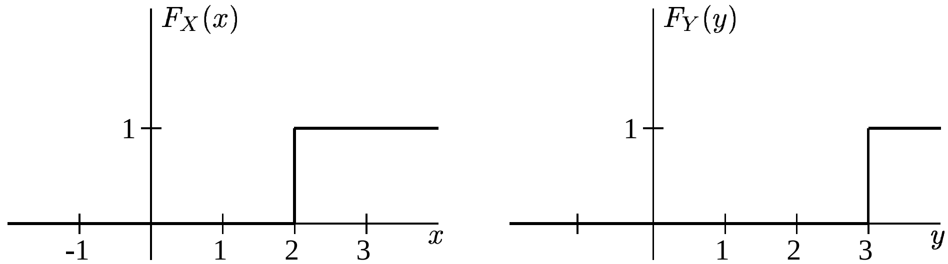
[5] (a) Find the value of the constant c.

[5] (b) Find $E[X]$.

[5] (c) Find the probability that $X^2 \geq \frac{1}{2}$.

[10] (d) Find $E[3X^2 + 4X + 2]$.

5. Suppose that your goal is to maximize the profit of your business. If you decide to travel to Xville, the profit (in dollars) for your business is a random variable X with cumulative distribution function $F_X(x)$ as given below. If you decide to travel to Yville, the profit (in dollars) for the trip is a random variable Y with cumulative distribution function $F_Y(y)$ as given below.



[7] (a) To which village would you travel to maximize your profit?

[8] (b) Let $Z = X^2 + Y^2$. Find the cumulative distribution function $F_Z(z)$.