ECE 603 - Probability and Random Processes, Fall 2008

Midterm Exam #1

October 15th, 6:00-8:00pm, Agricultural Engineering 119

Overview

- The exam consists of five problems for 130 points. The points for each part of each problem are given in brackets - you should spend your **two hours** accordingly.
- The exam is closed book, but you are allowed **one page-side** of notes. Calculators are not allowed. I will provide all necessary blank paper.

Testmanship

- **Full credit will be given only to fully justified answers.**
- Giving the steps along the way to the answer will not only earn full credit but also maximize the partial credit should you stumble or get stuck. If you get stuck, attempt to neatly define your approach to the problem and why you are stuck.
- If part of a problem depends on a previous part that you are unable to solve, explain the method for doing the current part, and, if possible, give the answer in terms of the quantities of the previous part that you are unable to obtain.
- Start each problem on a new page. Not only will this facilitate grading but also make it easier for you to jump back and forth between problems.
- If you get to the end of the problem and realize that your answer must be wrong, be sure to write "this must be wrong because ..." so that I will know you recognized such a fact.
- Academic dishonesty will be dealt with harshly - the *minimum penalty* will be an "F" for the course.

1. I have analyzed two *independent* experiments, Experiment 1 and Experiment 2, to arrive at two separate probability spaces: $(\Omega_1, \mathcal{A}_1, P_1)$ and $(\Omega_2, \mathcal{A}_2, P_2)$, where:

   $\Omega_1 = \{1, 2\}$, $\mathcal{A}_1 = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$, and $P_1(\cdot)$ is defined by $P_1(\phi) = 0$, $P_1(\{1\}) = 0.4$, $P_1(\{2\}) = 0.6$, $P_1(\Omega) = 1$.

   $\Omega_2 = \{2, 4\}$, $\mathcal{A}_2 = \{\phi, \{2\}, \{4\}, \{2, 4\}\}$, and $P_2(\cdot)$ is defined by $P_2(\phi) = 0$, $P_2(\{2\}) = 0.2$, $P_2(\{4\}) = 0.8$, $P_2(\Omega) = 1$.

   [10] (a) Are these valid probability spaces? *Be sure to tell me all of the conditions that you checked to arrive at your answer.*

   [10] (b) My boss asks me to define a combined experiment as follows: Perform Experiment 1 and remember the result; then, perform Experiment 2 and remember the result. Now, write in your notebook the *outcome* of the combined experiment as an ordered pair with the first entry equal to the result of Experiment 1 and and the second entry equal to the result of Experiment 2. For example, an outcome might be "(1,4)". Find $(\Omega, \mathcal{A}, P)$ for the combined experiment. Use a $\mathcal{A}$ that captures as many events as possible, and be sure to write out explicitly at least half of the events in $\mathcal{A}$.

   [10] (c) Alas, the boss is fickle and changes his mind. Now he asks: perform Experiment 1 and remember the result; then, perform Experiment 2 and remember the result. Now, write in your notebook the *outcome* of the combined experiment as a *random variable* X equal to the result of Experiment 1 times the result of Experiment 2. For example, an outcome might be "4" (which is 2×2). Find $(\Omega, \mathcal{A}, P)$ for the random variable X. *Hint: Feel free to define P using the integral of a function if this makes it easier to represent.*

   Now, your buddy in the modeling department comes to you with yet another experiment description: $(\Omega_3, \mathcal{A}_3, P_3)$, where
   $\Omega_3 = \{1, 2, 3\}$, $\mathcal{A}_3 = \{\phi, \{1\}, \{2, 3\}, \{1, 2, 3\}\}$, and $P_3(\cdot)$ is defined by $P(\phi) = 0, P(\{1\}) = 0.1, P(\{2, 3\}) = 0.9, P(\{1, 2, 3\}) = 1$.

   [5] (d) Is $(\Omega_3, \mathcal{A}_3, P_3)$ a valid probability space? *Be sure to tell me all of the conditions that you checked to arrive at your answer.*

   [5] (e) Your boss asks you to use the description of $(\Omega_3, \mathcal{A}_3, P_3)$ to find the probability that a "3" is observed. How do you respond?

2. Tell whether the following statements are "True" or "False". *If you answer "True", prove the result. If you answer "False", give a counterexample.*

   [5] (a) If the events A and B are *independent* with $P(A) = 0.1$ and $P(B) = 0.1$, then the events A and B *cannot* be disjoint (i.e. mutually exclusive).

   [5] (b) If the events C and D are *independent* with $P(C) = 0.1$ and $P(D) = 0.0$, then the events C and D *cannot* be disjoint (i.e. mutually exclusive).

   [10] (c) If the two events E and F are *independent*, then the events $\overline{E}$ and $\overline{F}$ are independent.

   [5] (d) If the two events G and H are *mutually exclusive* (i.e. disjoint), then it must be the case that $\overline{G}$

and $\overline{H}$ are mutually exclusive (i.e. disjoint).

[5] (e) For events J and K, if we know that $P(J) \leq 0.1$, then $P(J|K)$ must be ≤ 0.1.

[5] (f) For events L and M, if we know that $P(L) \leq 0.1$, then $P(L \bigcap M)$ must be ≤ 0.1.

3. A salesman visits one of three cities: X-ville, Y-ville, and Z-ville. When he visits a given city, the corresponding probability density function (pdf) of the money that he obtains is given by:

$$\begin{aligned} f_X(x) &= \frac{1}{4}\,\delta(x) + \frac{1}{2}\,\delta(x-5) + \frac{1}{4}\,\delta(x-10) \\ f_Y(y) &= \frac{1}{2}\,\delta(y) + \frac{1}{2}\,\delta(y-10) \\ f_Z(z) &= \begin{cases} 1/10, & 0 \leq z \leq 10 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

[8] (a) Suppose he chooses a city at random to visit. Find the probability that he makes greater than or equal to \$5.

[8] (b) Suppose he chooses a city at random to visit. Given that he makes greater than or equal to \$5, find the probability that he visited city X.

[8] (c) Any of the cities can claim that they are the "best" city for the salesman to obtain money - if they use the correct argument. Give the argument that each can make. In other words, for each city, give a measure by which it is the "best".

[6] (d) Suppose he does 20 visits to city Y, and the money obtained for each visit is indepedent of any other visit. Write an expression for the probability that he makes more than \$150. (*Your expression should only contain simple terms that are easily evaluated.*)

4. [15] I draw an ordered pair (x, y) at random (i.e. all points equally likely) from the unit square $[0,1] \times [0,1]$, and define the random variable $Z = x + y$. Find the cumulative distribution function $F_Z(z)$ and probability density function $f_Z(z)$.

5. A *continuous* random variable X has a cumulative distribution function (CDF) of the form:

$$F_X(x) = \begin{cases} 0, & x \leq -2 \\ A(1 + \sin(bx)), & -2 < x \leq 2 \\ 1, & x > 2 \end{cases}$$

[5] (a) Find the values of A and b that make this a valid cumulative distribution function (CDF).

[5] (b) Find the probability density function $f_X(x)$.