ECE 603 - Probability and Random Processes, Fall 2006

Midterm Exam #1

October 16th, 6:00-8:00pm, Marston 132

Overview

- The exam consists of six problems for 120 points. The points for each part of each problem are given in brackets - you should spend your **two hours** accordingly.
- The exam is closed book, but you are allowed **one page-side** of notes. Calculators are not allowed. I will provide all necessary blank paper.

Testmanship

- **Full credit will be given only to fully justified answers.**
- Giving the steps along the way to the answer will not only earn full credit but also maximize the partial credit should you stumble or get stuck. If you get stuck, attempt to neatly define your approach to the problem and why you are stuck.
- If part of a problem depends on a previous part that you are unable to solve, explain the method for doing the current part, and, if possible, give the answer in terms of the quantities of the previous part that you are unable to obtain.
- Start each problem on a new page. Not only will this facilitate grading but also make it easier for you to jump back and forth between problems.
- If you get to the end of the problem and realize that your answer must be wrong, be sure to write "this must be wrong because ..." so that I will know you recognized such a fact.
- Academic dishonesty will be dealt with harshly - the *minimum penalty* will be an "F" for the course.

Hint: You may find the following fact useful as you solve this exam:

$$P\left(\bigcap_{n=1}^{\infty}(x-\frac{1}{n},x+\frac{1}{n})\right)=\lim_{n\to\infty}P((x-\frac{1}{n},x+\frac{1}{n}))$$

1. [15] I flip a fair coin twice (assume the flips are independent) and record the outcome of these flips in order. For example, for "head followed by head", the outcome is "HH". Your job is to define a probability model that will used by one of your co-workers to analyze the experiment. You are not aware of what questions he/she might want to ask, so you want to generate as complete a model as possible (e.g. you do not want to use a trivial $\mathcal{F}$). Provide a probability space $(S,\mathcal{F},P)$ for this experiment. Since the size of the sets involved here is not that large, be explicit about how each of these three things are defined. In particular, write out all of the sets in $\mathcal{F}$ and give the probability of each.

2. Your buddy who works in the modeling department provides you with the following probability space: $S=(0,1)$, $\mathcal{F}=\mathcal{B}$ (where the Borel field is restricted to $(0,1)$, of course) and, for any interval,

$$P((a,b))=\begin{cases}\frac{b^2-a^2}{2}, & 0\le a<b\le\frac{1}{2}\\ \frac{b^2-a^2}{2} & \frac{1}{2}\le a<b\le 1\\ \frac{1}{2}+\frac{b^2-a^2}{2}, & a<\frac{1}{2}<b\le 1\end{cases}$$

   [5] (a) Find the probability of the outcome $\frac{1}{4}$.

   [10] (b) Find the probability of the set of irrational numbers in $(0,1)$.

3. The word "algebra" contains four consonants ("l", "g", "b", and "r") and three vowels ("a", "e", "a"). Suppose I place these seven letters in a bag along with an eighth symbol that is a "blank space".

   I draw the eight symbols out of the bag at random one at a time and place them left to right in the order drawn. The blank space will cause there to always be two words in the resulting expression (if the blank space is drawn first and thus appears at the left end, assume the first word is empty; if the blank space is drawn last and thus appears at the right end, assume the second word is empty).

   Answer each part of this problem independently.

   [8] (a) Suppose that the "blank space" is the third symbol drawn (and thus I have a two-letter word and a five letter word). What is the probability that both words have at least one vowel?

   [12] (b) **This part has nothing to do with part (a). Read the question again, skip over part (a), and answer this part.** What is the probability that each word will contain at least one consonant?

4. Alex, Brian, and Chris take turns rolling a fair six-sided die (in that order: Alex, Brian, Chris, Alex, Brian, Chris, Alex, . . .). The game stops when somebody rolls a "6", and the person rolling the "6" is declared the winner. Assume that rolls of the die are independent.

   [10] (a) Find the probability that Alex wins the game.

   [8] (b) Let X be the number of times Alex rolls the die in a given game. Find the probability that $X = n$ for $n = 0, 1, 2, 3, \ldots$.

   [7] (c) Suppose that Alex wins on *his* sixth roll. Let Y be the total number of 5's that have appeared on the die before that point. What is the probability that $Y = 5$?

5. The probability density function of a random variable X is given by $f_X(x)$, where:

$$f_X(x) = \begin{cases} cx^2, & -2 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

   [5] (a) Find the value of the constant c.

   [6] (d) Find the probability that $X^2 \geq 1$.

   [6] (e) Find the probability that $X - 1 \geq -\frac{1}{4}$.

   [13] (f) Let the random variable Y be defined by:

$$Y = \begin{cases} -X, & X \leq 0 \\ 0, & X \geq 0 \end{cases}$$

   Find the probability density function of Y.

6. Suppose that I am observing a network connection that is good ("G") with probability 0.9 and bad ("B") with probability 0.1. Let T (in seconds) be the time until the first packet arrives.

   If the connection is good, T has probability density function:

$$f_G(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & \text{else} \end{cases}$$

   If the connection is bad, T has probability density function:

$$f_B(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & \text{else} \end{cases}$$

   [7] (a) What is the probability that the first packet arrives during the first second?

   [8] (b) Given that the first packet arrives during the first second, what is the probability that the link is good?