ECE 603 - Probability and Random Processes, Fall 2002

Midterm Exam #1

October 17th, 6:00-8:00pm, Marston 132

Overview

- The exam consists of five problems for 120 points. The points for each part of each problem are given in brackets - you should spend your **two hours** accordingly.
- The exam is closed book, but you are allowed **one page-side** of notes. Calculators are not allowed. I will provide all necessary blank paper.

Testmanship

- **Full credit will be given only to fully justified answers.**
- Giving the steps along the way to the answer will not only earn full credit but also maximize the partial credit should you stumble or get stuck. If you get stuck, attempt to neatly define your approach to the problem and why you are stuck.
- If part of a problem depends on a previous part that you are unable to solve, explain the method for doing the current part, and, if possible, give the answer in terms of the quantities of the previous part that you are unable to obtain.
- Start each problem on a new page. Not only will this facilitate grading but also make it easier for you to jump back and forth between problems.
- If you get to the end of the problem and realize that your answer must be wrong, be sure to write "this must be wrong because . . . " so that I will know you recognized such a fact.
- Academic dishonesty will be dealt with harshly - the *minimum penalty* will be an "F" for the course.

1. A number is chosen at random from the interval [0,1]. As is the standard case, the probabilities are defined on the Borel σ-algebra (restricted to [0,1]). Starting from first principles (i.e. definition of the Borel σ-algebra, axioms of probability, etc.), answer the following three parts:

   [10] (a) Let A be a subset of $[0, 1]$ that is **not** in the Borel σ-algebra. Show that A must contain an uncountable number of elements.

   [5] (b) Let D be an arbitrary uncountable subset of $[0, 1]$. Is $\overline{D}$, the complement of D, necessarily countable?

   [10] (c) Let C be the set of irrational numbers in $[0, 1]$; that is, $x \in C$ if and only if $0 \leq x \leq 1$ and $x \neq \frac{m}{n}$ for all $m \in \{0, 1, 2, 3, \ldots\}$, $n \in \{0, 1, 2, 3, \ldots\}$. Find the probability of C.

2. [10] A number is chosen from the interval [0,1] such that the likelihood of a given result x is proportional to the value x. Define a non-trivial probability space for this experiment; that is, find $(\Omega, \mathcal{A}, P)$, where Ω is the observation space, $\mathcal{A}$ is a set of subsets of Ω to which probabilities are assigned, and P is a probability mapping from $\mathcal{A}$ to $[0, 1]$.

3. Let the continuous random variable X be uniformly distributed between 0 and 1. Let the discrete random variable Z have probability mass function defined by $P(Z = 1) = \frac{1}{2}$, $P(Z = 2) = \frac{1}{3}$, and $P(Z = 3) = \frac{1}{6}$. Form the random variable $Y = XZ$. Assume X and Z are independent.

   [10] (a) Given that $Y < \frac{1}{5}$, what is the probability that $Z = 2$.

   [10] (b) Find $f_Y(y)$, the probability density function of Y.

   [10] (c) Find $E[Y]$ and Var$[Y]$.

4. The random variables X and Y have joint probability density function:

$$f_{X,Y}(x,y) = \begin{cases} c(x+1), & 0 < Y < X < 1 \\ 0, & \text{else} \end{cases} .$$

[5] (a) Compute the constant c.

[5] (b) Compute the marginal probability density function $f_X(x)$.

[5] (c) Compute the conditional probability density function $f_{Y|X}(y|x)$.

[10] (d) Compute $E[Y|X = x]$ and Var$[Y|X = x]$. Interpret your answers at $x = 0$.

[5] (e) Let $W = \frac{1}{X+1}$. Find $E[W]$.

[10] (f) Let $Z = XY$. Compute $F_Z(z)$, the cumulative distribution function (CDF) of Z.

5. [15] Let the random variables $\{X_i, i = 1, 2, \ldots, N\}$ be mutually independent and each have probability density function:

$$f_X(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases} .$$

and define Y as the maximum of X_1, X_2, …, X_N. Find $f_Y(y)$, the probability density function of Y.