1) (a)

$$P(X_1 = 1) = P(\omega = \text{Apple}) = 1/5$$
$$P(X_1 = 2) = P(\omega = \text{Banana}) = 1/5$$
$$P(X_1 = 3) = P(\omega = \text{Lime}) = 1/5$$
$$P(X_1 = 4) = P(\omega = \text{Pear}) = 1/5$$
$$P(X_1 = 5) = P(\omega = \text{Orange}) = 1/5$$

$$f_{X_1}(x) = 1/5\,\delta(x) + 1/5\,\delta(x-1) + 1/5\,\delta(x-2) + 1/5\,\delta(x-3) + 1/5\,\delta(x-4)$$

(b)

$\{X_n\} \to 0$ in every way, which I can get by showing:

$\{X_n\} \xrightarrow{a.s.} 0$: For any ω, $X_n(\omega) \to 0$; thus,

$$P(\{\omega : X_n(\omega) \to 0\}) = 1$$

$\{X_n\} \xrightarrow{m.s.} 0$: $E[|X_n - X|^2] = E[|X_n|^2]$

$$\leq E[|5/n|^2] = 25/n^2 \to 0$$

2) a)

Find the distribution of X_n

$$F_{X_n}(x) = P(X_n \le x), \qquad 0 \le x \le 1$$

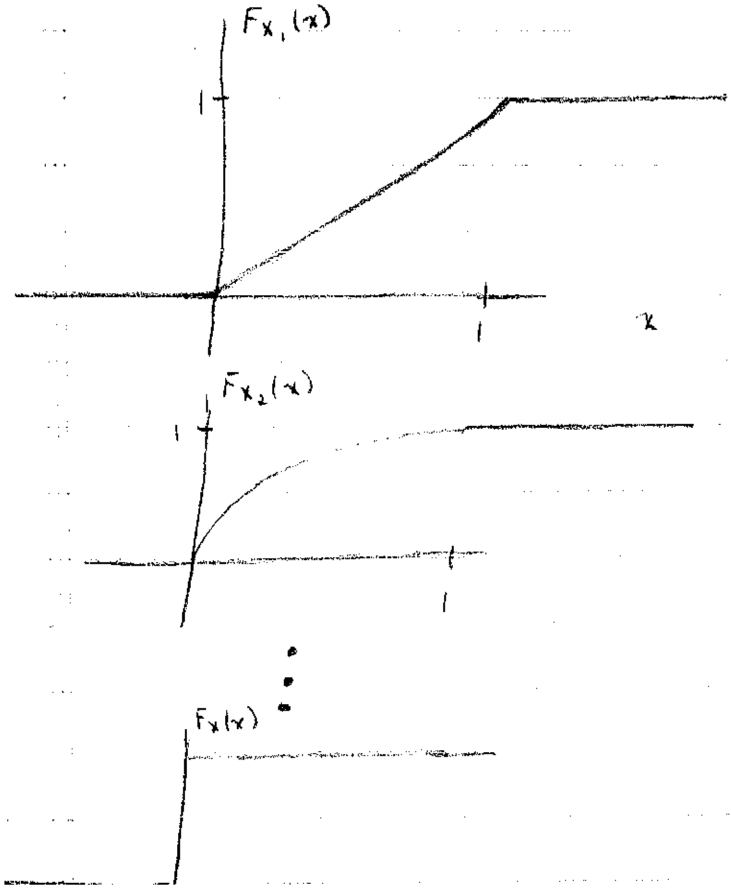
$$= P(\omega^n \le x)$$

$$= P(\omega \le x^{1/n})$$

$$= \int_0^{x^{1/n}} du$$

$$= x^{1/n}$$

$$\Rightarrow F_{X_n}(x) = \begin{cases} 0, & x \le 0 \\ x^{1/n}, & 0 < x \le 1 \\ 1, & x > 1 \end{cases}$$



Need to show it converges at every "continuity" point of $F_X(x)$ — all except $x = 0$, in this case.

$x < 0$: trivial

$x > 1$: trivial

$0 < x \le 1$:

$$\lim_{n \to \infty} x^{1/n} = 1$$

Thus,

$$X_n \xrightarrow{D} X = 0$$

(b)

$$P(|X_n - X| \geq \varepsilon) = P(X_n \geq \varepsilon)$$
$$= 1 - P(X_n < \varepsilon)$$
$$= 1 - F_{X_n}(\varepsilon)$$
$$= 1 - \varepsilon^{1/n} \to 0 \quad \text{as } n \to \infty$$

Thus, $X_n \xrightarrow{P} X = 0$

(c) $$E[|X_n - X|^2] = E[X_n^2]$$
$$= \int_{-\infty}^{\infty} x^2 f_{X_n}(x)\,dx$$

$$f_{X_n}(x) = d/dx\, F_{X_n}(x) = \begin{cases} 1/n\, x^{1/n - 1}, & 0 \leq x \leq 1 \\ 0, & \text{else} \end{cases}$$

$$= \int_0^1 1/n\, x^{1/n + 1}\,dx$$
$$= 1/n \frac{1}{1/n + 2} x^{1/n + 2}\Big|_0^1$$
$$= \frac{1}{2 + n} \to 0 \quad \text{as } n \to \infty$$

Thus, $X_n \xrightarrow{m.s.} X = 0$

(d) Consider any $\omega_0 \in [0,1)$
$$X_n(\omega_0) = \omega_0^n \to 0 \quad \text{as } n \to \infty$$

and $P([0,1)) = 1$. Thus, $\exists$ a set $A = [0,1)$ s.th. $X_n(\omega_0) \to X(\omega_0)$ $\forall \omega_0 \in A$ and $P(A) = 1$.

Thus, $X_n \xrightarrow{a.s.} X = 0$

3)

(a) $X_n(\omega) = (\omega + 1/n)^2$

- $X_1(\omega) = (\omega + 1)^2$

  $X_2(\omega) = (\omega + 1/2)^2$

Note that $X_1(\omega)$ determines $X_2(\omega)$!

$$\omega = \sqrt{X_1(\omega)} - 1$$

$$\Rightarrow X_2(\omega) = \left(\sqrt{X_1(\omega)} - 1 + 1/2\right)^2 = \left(\sqrt{X_1(\omega)} - 1/2\right)^2$$

$$\Rightarrow f_{X_2|X_1}(x_2|x_1) = \delta\left(x_2 - (\sqrt{x_1} - 1/2)^2\right)$$

(b) I claim it goes in all ways to $X(\omega) = \omega^2$

Consider any $\omega \in (0,1)$

$$X_n(\omega) = \omega^2 + 2\omega/n + 1/n^2 \to \omega^2$$

Hence, $X_n(\omega) \to \omega^2$ pointwise $\Rightarrow X_n \xrightarrow{a.s.} X$

$X_n \xrightarrow{P} X$

$X_n \xrightarrow{D} X$

$$E[|X_n - X|^2] = E[|\omega^2 + 2\omega/n + 1/n^2 - \omega^2|^2]$$

$$= E[|2\omega/n + 1/n^2|^2] \leq E[(2/n + 1/n^2)^2]$$

$$= 4/n^2 + 4/n^3 + 1/n^4 \to 0 \Rightarrow X_n \xrightarrow{m.s.} X$$

(b)

- $X_1(n) = -w$

Per Exam 2, this flips around the pdf of w. But that pdf is symmetric

$$\Rightarrow f_{X_1}(x) = \begin{cases} 1, & -1/2 \le x \le 1/2 \\ 0, & \text{else} \end{cases}$$

- I claim $X_n \xrightarrow{D} X(w) = w$, but no other way

First, note that $f_{X_n(w)}(x)$ never changes $\Rightarrow F_{X_n}(x)$ never changes $\Rightarrow F_X(x) \to F_X(x)$, where $X(w) = w$.

Next, consider $X_n \xrightarrow{P} X(w) = w$. Let $\varepsilon = 1/2$.

<u>For n odd:</u>

$$P(|X_n - X| > 1/2) = P(|2w| > 1/2)$$

$$= P(|w| > 1/4)$$

$$= 1/2$$

Thus $P(|X_n - X| > \varepsilon)$ does not go to zero for $\varepsilon = 1/2$

$X \not\xrightarrow{P} X$, $X_n \not\xrightarrow{a.s.} X$, $X \not\xrightarrow{m.s.} X$

(c)

Looks like it is going to $X = 0$.

$$E[|X_n - 0|^2] = E[X_n^2] = 0 + \sum_{i=1}^{n} 1/n^2 \cdot i^2$$

$$= 1/n^2 \sum_{i=1}^{n} i^2$$

$$\geq 1/n^2 \cdot n^2 = 1$$

not in m.s.

For any $\epsilon > 0$:

$$P(|X_n - 0| > \epsilon) = P(|X_n| > \epsilon) = 1/n \to 0 \quad \text{as} \quad n \to \infty$$

$$X_n \xrightarrow{P} 0$$

$$X_n \xrightarrow{D} 0$$

4)

(a) Converges in all ways to $X=0$.

For any ω, $\omega^3/\sqrt{n} \to 0$ as $n \to \infty$; thus, $X_n \xrightarrow{a.s.} X$

$\Rightarrow X_n \xrightarrow{P} X$

$\Rightarrow X_n \xrightarrow{D} X$

$E\left[\left|\omega^3/\sqrt{n} - 0\right|^2\right] = E\left[\omega^6/n\right] = 1/n \int_0^1 \omega^6 \, d\omega = 1/7n \to 0$

$\Rightarrow X_n \xrightarrow{m.s.} X$

(b)

Converges in all ways to $X=0$.

For any irrational ω, $\omega/n \to 0$ as $n \to \infty$. Since $P(\overline{\mathbb{Q}}) = 1$ for this $P(\cdot)$, $X_n \xrightarrow{a.s.} X = 0$

$\Rightarrow X_n \xrightarrow{P} X = 0$

$\Rightarrow X_n \xrightarrow{\mathcal{D}} X = 0$

$$|X_n - X|^2 = \begin{cases} \omega^2/n^2, & \omega \text{ irrational} \\ 1, & \omega \text{ rational} \end{cases}$$

$\Rightarrow E[|X_n - X|^2] = \omega^2/n^2 \cdot 1 + 1 \cdot 0 = \omega^2/n^2 \to 0$ as $n \to \infty$

$X_n \xrightarrow{m.s.} X = 0$

(c) It doesn't go to $X=0$ in any way.

Since $P(\omega = 1/4) = 1/2$ $P(X_n = 1) \geq 1/2$ for any n

Hence $F_{X_n}(1/2) = P(X \leq 1/2) \leq 1/2$

but $F_X(1/2) = 1$; thus, $F_{X_n}(x) \not\to F_X(x)$ at continuity point $x = 1/2$

~~Thus, $X_n \xrightarrow{D} X$ (and, thus, in any other way)~~

(part (d) is on the next page)

~~(e) $X_n(\omega)$ is uniform on [0,1] for any n.~~

Thus, $X_n(\omega) \xrightarrow{D} X$ uniform on [0,1]

But it does not converge any other way.

Consider $\varepsilon = 0.01$

$$P(|X_{n+1} - X_n| < 2\varepsilon) = P(1/2 - \varepsilon \leq \omega \leq 1/2 + \omega) = 2\varepsilon.$$

for any n.

Thus, there does not exist X s.th.

$\lim_{\substack{n\to\infty \\ n \text{ odd}}} P(|X_n - X| < \varepsilon) \to 0$ — thus $X_n \not\xrightarrow{P} X$

and

$\lim_{\substack{n\to\infty \\ n \text{ even}}} P(|X_n - X| < \varepsilon) \to 0$ — $\Rightarrow X_n \not\xrightarrow{a.s.} X$

$\Rightarrow X_n \not\xrightarrow{m.s.} X$

(d) $X_n(\omega) \to \omega$ in all ways

For any ω, $|X_n(\omega) - \omega| \le 1/n$

$\Rightarrow X_n \xrightarrow{a.s.} X$

$\Rightarrow X_n \xrightarrow{P} X$

$\Rightarrow X_n \xrightarrow{D} X$

$E[|X_n(\omega)-\omega|^2] \le E[1/n^2] = 1/n^2 \to 0$ as $n \to \infty$

$\Rightarrow X_n \xrightarrow{m.s.} X$