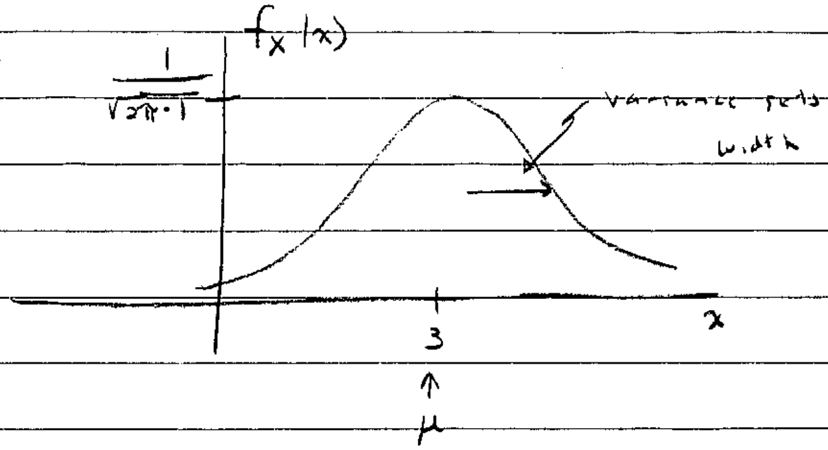
Homework #5 Solutions

ECE 603 Fall 2014

1)

(a)



(b)

$$P(-4 \le X \le 1) = \underbrace{P(X \le 1)}_{\frac{x-\mu}{\sqrt{2}\sigma} = -\sqrt{2}} - \underbrace{P(X \le -4)}_{\frac{x-\mu}{\sqrt{2}\sigma} = \frac{-7}{\sqrt{2}}}$$

$$= \frac{1}{2} - \frac{1}{2}\operatorname{erf}(\sqrt{2}) - \left(\frac{1}{2} - \frac{1}{2}\operatorname{erf}(7/\sqrt{2})\right)$$

$$= 1/2 \operatorname{erf}(7/\sqrt{2}) - 1/2 \operatorname{erf}(\sqrt{2})$$

(c) $P(X^2 \ge 10) = P(\{X \ge \sqrt{10}\} \cup \{X \le -\sqrt{10}\})$

$$\overset{\text{disjoint}}{=} P(X \ge \sqrt{10}) + P(X \le -\sqrt{10})$$

$$= 1 - \underbrace{P(X \le \sqrt{10})}_{\frac{x-\mu}{\sqrt{2}\sigma} = \frac{\sqrt{10}-3}{\sqrt{2}}} + \underbrace{P(X \le -\sqrt{10})}_{\frac{x-\mu}{\sqrt{2}\sigma} = \frac{-\sqrt{10}-3}{\sqrt{2}}}$$

$$= 1 - \left(\frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{\sqrt{10}-3}{\sqrt{2}}\right)\right) + \left(\frac{1}{2} - \frac{1}{2}\operatorname{erf}\left(\frac{-\sqrt{10}-3}{\sqrt{2}}\right)\right)$$

$$= 1 - \frac{1}{2}\operatorname{erf}\left(\frac{\sqrt{10}-3}{\sqrt{2}}\right) - \frac{1}{2}\operatorname{erf}\left(\frac{+\sqrt{10}+3}{\sqrt{2}}\right)$$

2)

$$P(Y \le y) = 1 - P(Y \ge y)$$

$$= 1 - \int_y^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(u-\mu)^2/2\sigma^2}\, du$$

$v = u - \mu$

$$\Rightarrow 1 - \int_{y-\mu}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-v^2/2\sigma^2}\, dv$$

Need: $v^2/2\sigma^2 = u^2/5$

$\Rightarrow u = v/\sqrt{\frac{2}{5}\sigma^2}$

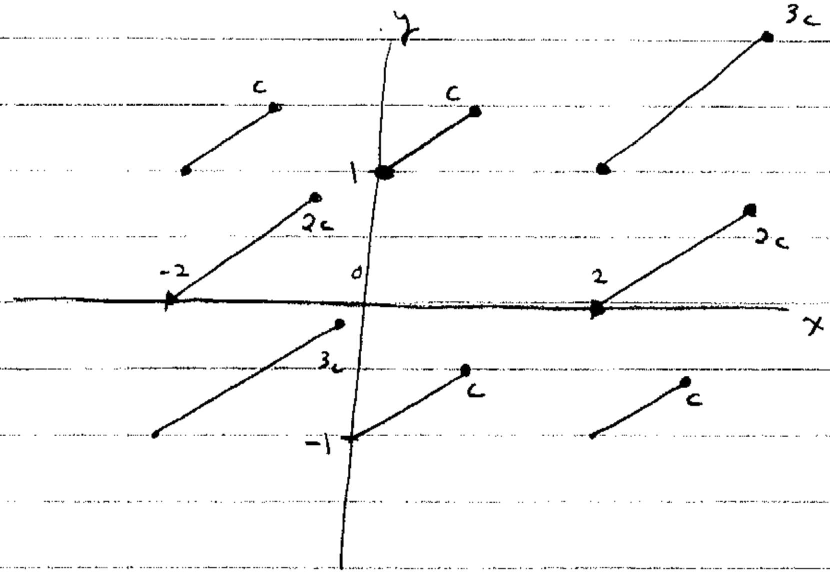
$$\left.\begin{aligned} u &= v/\sqrt{\tfrac{2}{5}\sigma^2} \\ du &= 1/\sqrt{\tfrac{2}{5}\sigma^2}\, dv \end{aligned}\right\} \Rightarrow 1 - \int_{\frac{y-\mu}{\sqrt{\frac{2}{5}\sigma^2}}}^{\infty} \frac{\sqrt{\frac{2}{5}\sigma^2}}{\sqrt{2\pi\sigma^2}} e^{-u^2/5}\, du$$

$$= 1 - \frac{2}{\sqrt{5\pi}} \int_{\frac{y-\mu}{\sigma}\cdot\sqrt{\frac{5}{2}}}^{\infty} \frac{1}{2} e^{-u^2/5}\, du$$

$$= \begin{cases} 1 - \dfrac{2}{\sqrt{5\pi}}\, \mathcal{Y}\left(\dfrac{y-\mu}{\sigma}\sqrt{\dfrac{5}{2}}\right), & y - \mu \ge 0 \\ \dfrac{2}{\sqrt{5\pi}}\, \mathcal{Y}\left(\dfrac{\mu - y}{\sigma}\sqrt{\dfrac{5}{2}}\right), & y - \mu < 0 \end{cases}$$

3)

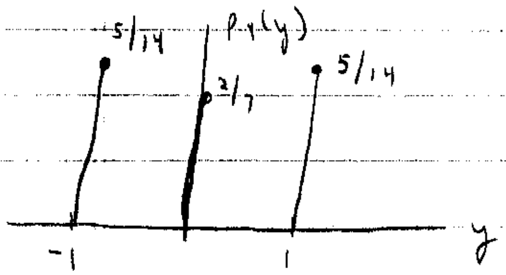
(a)



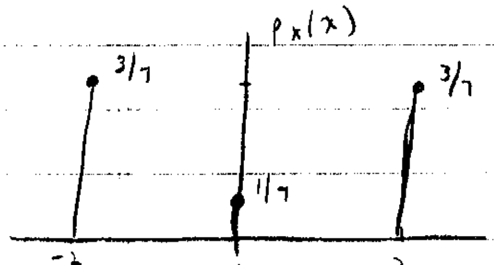
$$\sum_{(x,y)\in S} p_{X,Y}(x,y) = 14c = 1 \Rightarrow c = 1/14$$

(b)

$$p_Y(y) = \sum_x p_{X,Y}(x,y) = \begin{cases} 5/14, & y=-1 \\ 2/7, & y=0 \\ 5/14, & y=1 \\ 0, & \text{else} \end{cases}$$



(c) $$p_X(x) = \sum_y p_{X,Y}(x,y) = \begin{cases} 3/7, & x=-2 \\ 1/7, & x=0 \\ 3/7, & x=2 \\ 0, & \text{else} \end{cases}$$



(d)

$$E[XY] = \sum_{(x,y)\in S} xy\, p_{X,Y}(x,y)$$

$$= \underbrace{3/14\,(2)}_{x=2,\,y=1} + \underbrace{1/14\,(-2)}_{x=-2,\,y=1} + \underbrace{3/14\,(2)}_{x=-2,\,y=-2} + \underbrace{1/14\,(-2)}_{x=2,\,y=-1}$$

$$= 4/7$$

(e) $$P(Y < X) = \sum_{\substack{(x,y) \text{ s.th.} \\ y<x}} p_{X,Y}(x,y) = \underbrace{1/14}_{x=0,\,y=-1} + \underbrace{1/14}_{x=2,\,y=-1} + \underbrace{2/14}_{x=2,\,y=0} + \underbrace{3/14}_{x=2,\,y=1} = 1/2$$

(f) $$P(Y=X) = \sum_{\substack{(x,y) \text{ s.th.} \\ y=x}} p_{X,Y}(x,y) = \underbrace{0}_{x=0,\,y=0}$$

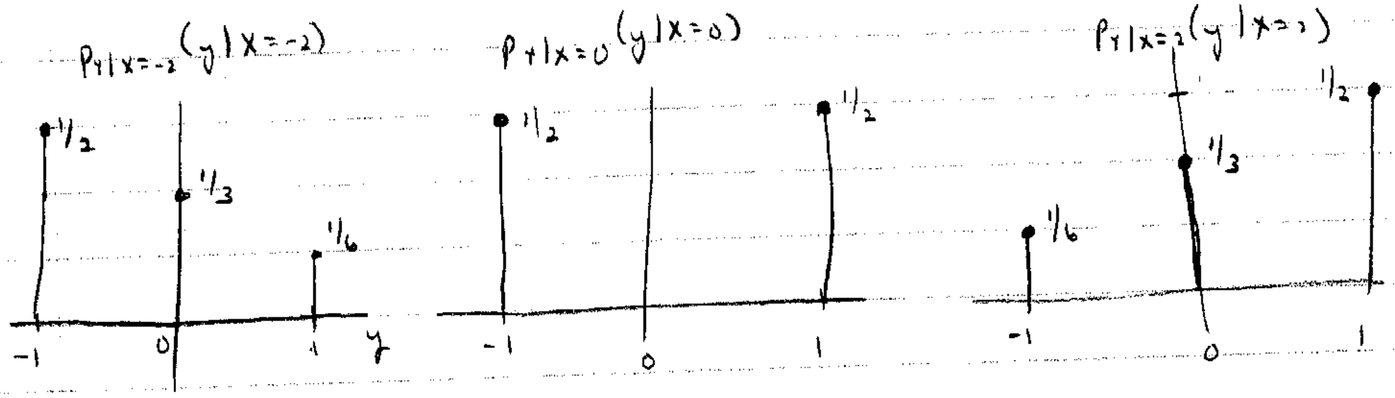
you didn't have to do this, but look

I will give the values of $F_{X,Y}(x,y)$ in each region (think "everything down and left"):

	3/7	4/7	1
0	5/14	3/7	9/14
0	3/14	2/7	5/14
	0	0	

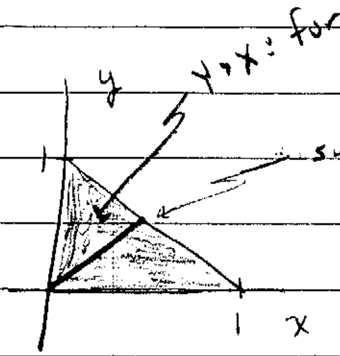
(g)

$$p_{Y|X=x}(y \mid X=x) = \frac{p_{X,Y}(x,y)}{p_X(x)}$$



(h) No. $p_{Y|X=-2}(y \mid X=-2) \neq p_Y(y)$.

4)



(a) $$\int_0^1 \int_0^{1-y} c(x+y)\,dx\,dy$$

$$= c\int_0^1 \left(x^2/2 + xy\right)\Big|_0^{1-y} dy$$

$$= c\int_0^1 \left((1-y)^2/2 + (1-y)y\right) dy$$

$$= c\int_0^1 \left(1/2 - \cancel{y} + y^2/2 + \cancel{y} - y^2\right) dy$$

$$= c\int_0^1 \left(1/2 - y^2/2\right) dy$$

$$= c\left(1/2 - 1/6\right) = 1 \quad \Rightarrow \quad c = 3$$

(b) $$f_X(x) \underset{\substack{\uparrow \\ 0 \le x \le 1}}{=} \int_0^{1-x} 3(x+y)\,dy = \left(3xy + 3/2\, y^2\right)\Big|_0^{1-x}$$

$$= 3x(1-x) + 3/2(1-2x+x^2)$$

$$= \cancel{3x} - 3x^2 + 3/2 - \cancel{3x} + 3/2\, x^2$$

$$= \begin{cases} 3/2 - 3/2\, x^2, & 0 \le x \le 1 \\ 0, & \text{else} \end{cases}$$

Note: $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$ ✓

$$f_Y(y) \underset{\substack{\uparrow \\ 0 \le y \le 1}}{=} \int_0^{1-y} 3(x+y)\,dx \underset{\substack{\uparrow \\ \text{by symmetry}}}{=} \begin{cases} 3/2 - 3/2\, y^2, & 0 \le y \le 1 \\ 0, & \text{else} \end{cases}$$

(d)

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$= \begin{cases} \frac{\cancel{3}(x+y)}{\cancel{3}/2 - 3/2\, x^2} = \frac{2(x+y)}{1-x^2}, & 0 \le x \le 1,\ 0 \le y \le 1-x \\ 0, & \text{else} \end{cases}$$

<u>Check:</u> $$\int_0^{1-x} \frac{2(x+y)}{1-x^2}\, dy = \frac{1}{1-x^2}\left(2xy + y^2\right)\Big|_0^{(1-x)}$$

$$= \frac{1}{1-x^2}\left(2x(1-x) + 1 - 2x + x^2\right)$$

$$= \frac{1}{1-x^2}\left(\cancel{2x} - 2x^2 + 1 - \cancel{2x} + x^2\right)$$

$$= 1$$ <u>Check</u> ✓

(e)
$$P(Y > X) = \int_0^{1/2} \int_x^{1-x} 3(x+y)\, dy\, dx$$

$$= \int_0^{1/2} \left(3xy + 3/2\, y^2\right)\Big|_x^{1-x} dx$$

$$= \int_0^{1/2} \left(\cancel{3x} - 3x^2 + 3/2\left(1 - \cancel{2x} + \cancel{x^2}\right) - 3x^2 - \cancel{3/2\, x^2}\right) dx$$

$$= \int_0^{1/2} \left(3/2 - 6x^2\right) dx$$

$$= \left(3/2\, x - 2x^3\right)\Big|_0^{1/2} = 3/4 - 2/8 = 1/2$$

↑ also, suggested by symmetry

(b)

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^1 x\left(3/2 - 3/2\,x^2\right)dx = \int_0^1 \left(3/2\,x - 3/2\,x^3\right)dx$$

$$= \left(3/4\,x^2 - 1/2\,x^3\right)\Big|_0^1$$

$$= 1/4$$

$$Var[X] = E[X^2] - (E[X])^2$$

$$E[X^2] = \int_0^1 x^2\left(3/2 - 3/2\,x^2\right)dx = \int_0^1 \left(3/2\,x^2 - 3/2\,x^4\right)dx$$

$$= \left(1/2\,x^3 - 3/10\,x^5\right)\Big|_0^1 = 1/5$$

$$\Rightarrow Var[X] = 1/5 - (1/4)^2 = 7/60$$

e)

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^1 x\left(\tfrac{3}{2} - \tfrac{3}{2}x^2\right)dx = \int_0^1 \left(\tfrac{3}{2}x - \tfrac{3}{2}x^3\right)dx$$

$$= \left(\tfrac{3}{4}x^2 - \tfrac{3}{8}x^4\right)\Big|_0^1$$

$$= 3/8$$

$$Var(X) = E[X^2] - (E[X])^2$$

$$E[X^2] = \int_0^1 x^2\left(\tfrac{3}{2} - \tfrac{3}{2}x^2\right)dx = \int_0^1 \left(\tfrac{3}{2}x^2 - \tfrac{3}{2}x^4\right)dx$$

$$= \left(\tfrac{1}{2}x^3 - \tfrac{3}{10}x^5\right)\Big|_0^1 = 1/5$$

$$\Rightarrow Var(X) = (1/5) - (3/8)^2$$

$$= 1/5 - 9/64 = 64/320 - 45/320 = 19/320$$

(f)

Note Y has the same marginal pdf as X

$$\Rightarrow E[Y] = E[X] = 1/4 \qquad Var[Y] = Var[X] = 7/60$$

(g)

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_0^1 \int_0^{1-y} (3x^2y + 3xy^2)\,dx\,dy$$

$$= \int_0^1 \left(x^3 y + 3/2\, x^2 y^2\right)\Big|_0^{1-y} dy$$

$$= \int_0^1 \left((1 - 3y + 3y^2 - y^3)y + 3/2(1 - 2y + y^2)y^2\right) dy$$

$$= \int_0^1 \left(y - 3y^2 + 3y^3 - y^4 + 3/2\, y^2 - 3y^3 + 3/2\, y^4\right) dy$$

$$= \int_0^1 \left(y - 3/2\, y^2 + 1/2\, y^4\right) dy$$

$$= \left(1/2\, y^2 - 1/2\, y^3 + 1/10\, y^5\right)\Big|_0^1 = 1/10$$

$$cov(X,Y) = E[XY] - E[X]E[Y] = 3/80$$

(h) $E[X+Y] = E[X] + E[Y] = 1/2$

(i) $Var[X+Y] = Var(X) + Var(Y) + 2\,cov(X,Y)$

$$= 7/60 + 7/60 + 3/40 = 40/120 = 1/3$$

5)

(a)

$$E[XY] \overset{\text{independence}}{=} E[X]E[Y] = \int_{-\infty}^{\infty} x f_X(x)\,dx \int_{-\infty}^{\infty} y f_Y(y)\,dy$$

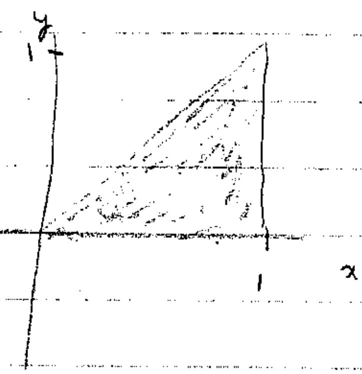
$$= \int_0^1 x\, 2x\, dx \int_0^1 y\, 3y^2\, dy$$

$$= \left(\tfrac{2}{3} x^3 \Big|_0^1\right)\left(\tfrac{3}{4} y^4 \Big|_0^1\right)$$

$$= 1/2$$

(b) X, Y independent $\Rightarrow$ X, Y uncorrelated $\Rightarrow$ $\text{cov}(X,Y) = 0$

(c) $$P(X>Y) = \int_0^1 \int_0^x f_{X,Y}(x,y)\,dy\,dx$$



$$\overset{\text{independence}}{=} \int_0^1 \int_0^x f_X(x) f_Y(y)\,dy\,dx$$

$$= \int_0^1 \int_0^x 6xy^2\,dy\,dx$$

$$= \int_0^1 2x\, y^3 \Big|_0^x\, dx$$

$$= \int_0^1 2x^4\, dx = 2/5$$

6)

Z is a linear combination of jointly Gaussian X and Y.

$\Rightarrow Z \sim N(\mu_Z, \sigma_Z^2)$

$$\mu_Z = E[2X - 3Y]$$
$$= 2E[X] - 3E[Y] = -1$$

$$\sigma_Z^2 = E[Z^2] - (E[Z])^2$$

$$E[Z^2] = E[(2X - 3Y)^2]$$
$$= E[4X^2 - 12XY + 9Y^2]$$
$$= 4E[X^2] - 12E[XY] + 9E[Y^2] = \cancel{4 \cdot 3} - 4\sqrt{14} - \cancel{12} + 72$$
$$= 72 - 4\sqrt{14}$$

$$Var(X) = E[X^2] - (E[X])^2 = 3 - 1^2 = 2$$

$$Var(Y) = E[Y^2] - (E[Y])^2 = 8 - 1^2 = 7$$

$$1/3 = \frac{E[XY] - E[X]E[Y]}{\sqrt{Var(X)Var(Y)}} = \frac{E[XY] - 1}{\sqrt{14}}$$

$$E[XY] = \sqrt{14}/3 + 1$$

$$\Rightarrow \sigma_Z^2 = 71 - 4\sqrt{14}$$

$$P(Z > 3) = 1 - P(Z \leq 3)$$

$$\frac{x - \mu}{\sqrt{2}\sigma} = \frac{3 + 1}{\sqrt{2(71 - 4\sqrt{14})}} = \frac{2\sqrt{2}}{\sqrt{71 - 4\sqrt{14}}} > 0$$

$$= 1 - \left(1/2 + \frac{1}{2} erf\left(\frac{2\sqrt{2}}{\sqrt{71 - 4\sqrt{14}}}\right)\right) = \frac{1}{2} - \frac{1}{2} erf\left(\frac{2\sqrt{2}}{\sqrt{71 - 4\sqrt{14}}}\right)$$