Homework #2 Solutions

ECE 603 Fall, 2016

1) Yes

Let $A \in \mathcal{A}_1 \cap \mathcal{A}_2$

$\Rightarrow A \in \mathcal{A}_1, A \in \mathcal{A}_2$

$\Rightarrow \bar{A} \in \mathcal{A}_1, \bar{A} \in \mathcal{A}_2$ ($\mathcal{A}_1, \mathcal{A}_2$ are algebras)

$\Rightarrow \bar{A} \in \mathcal{A}_1 \cap \mathcal{A}_2$ (closed under complementation)

Let $A, B \in \mathcal{A}_1 \cap \mathcal{A}_2$

$\Rightarrow A, B \in \mathcal{A}_1, A, B \in \mathcal{A}_2$

$\Rightarrow A \cup B \in \mathcal{A}_1, A \cup B \in \mathcal{A}_2$ ($\mathcal{A}_1, \mathcal{A}_2$ are algebras)

$\Rightarrow A \cup B \in \mathcal{A}_1 \cap \mathcal{A}_2$ (closed under unions)

(b)

No

Let $S = \{1, 2, 3, 4, 5\}$

$\mathcal{A}_1 = \{\phi, S, \{1,2\}, \{3,4,5\}\}$

$\mathcal{A}_2 = \{\phi, S, \{1,2,3\}, \{4,5\}\}$

both are algebras

Now, $\mathcal{A}_1 \cup \mathcal{A}_2 = \{\phi, S, \{1,2\}, \{1,2,3\}, \{4,5\}, \{3,4,5\}\}$

and $\{1,2\} \cup \{4,5\} = \{1,2,4,5\} \notin \mathcal{A}_1 \cup \mathcal{A}_2$

not closed under unions.

2)

$(S, \mathcal{A}, P)$ given by

$S = \{HH, TH, HT, TT\}$

	P
$\mathcal{A} = \{ \phi$	0
$\{HH\},$	$1/4$
$\{TH\},$	$1/4$
$\{HT\},$	$1/4$
$\{TT\},$	$1/4$
$\{HH, TH\},$	$1/2$
$\{HH, HT\},$	$1/2$
$\{HH, TT\},$	$1/2$
$\{TH, HT\},$	$1/2$
$\{TH, TT\},$	$1/2$
$\{HT, TT\},$	$1/2$
$\{TH, HT, TT\},$	$3/4$
$\{HH, HT, TT\},$	$3/4$
$\{HH, TH, HT\},$	$3/4$
$S \}$	1

3) (a)

$$1 = P((0,1)) = 1/2 + c \cdot ((1-0) - 1/2(1^2 - 0^2))$$

$$= 1/2 + c \cdot (1/2)$$

$$\Rightarrow c = 1$$

(b)

$$P(X = 1/4) = P\left(\bigcap_{n=1}^{\infty} (1/4 - 1/n, 1/4 + 1/n)\right)$$

$$= P\left(\bigcap_{n=5}^{\infty} (1/4 - 1/n, 1/4 + 1/n)\right)$$

$$= \lim_{n \to \infty} P((1/4 - 1/n, 1/4 + 1/n))$$

$$= \lim_{n \to \infty} \left(((1/4 + 1/n) - (1/4 - 1/n)) - 1/2((1/4 + 1/n)^2 - (1/4 - 1/n)^2) \right)$$

$$= \lim_{n \to \infty} \left(2/n - 1/2 (4/n) \right)$$

$$= 0$$

(c) $P(\bar{Q}) = 1 - P(Q)$

$$= 1 - P\left(\bigcup_{x \in Q} \{x\}\right)$$

$$= 1 - \sum_{x \in Q} P(\{x\})$$

$= 1 - 1/2 \Leftarrow$ all have probability zero except $P(X = 1/2) = 1/2$

$$= 1/2$$

Law of Total Probability $\qquad Q_1, \bar{Q}_1$ a partition

4(a) $P(Q_2) = P(Q_2 | Q_1)\, P(Q_1) + P(Q_2 | \bar{Q}_1)\, P(\bar{Q}_1)$

$= 0.9 \cdot 0.9 + 0.3 \cdot 0.1 = 0.84$

(b) $P(Q_1 | Q_2) \overset{\text{Bayes Rule}}{=} \dfrac{P(Q_2 | Q_1)\, P(Q_1)}{P(Q_2)} = \dfrac{0.9 \cdot 0.9}{0.84} = \dfrac{0.81}{0.84}$

(c) $P(Q_1 | Q_3) \overset{\text{independent}}{=} P(Q_1) = 0.9$

(d) • Need Q_1 and (Q_2 or Q_3)

$A = Q_1 \cap (Q_2 \cup Q_3)$

• $P(A) = ((Q_1 \cap Q_2) \cup (Q_1 \cap Q_3))$ $\qquad Q_1 \cap Q_2 \cap Q_3$

$= P(Q_1 \cap Q_2) + P(Q_1 \cap Q_3) - P(\cancel{(Q_1 \cap Q_2)} \cap (Q_1 \cap Q_3))$

$= P(Q_2 | Q_1)\, P(Q_1) + P(Q_1)\, P(Q_3) - P(Q_3)\, P(Q_2 | Q_1)\, P(Q_1)$

$= 0.9 \cdot 0.9 + 0.9 \cdot 0.5 - 0.5 \cdot 0.9 \cdot 0.9$

$= 0.81 + 0.45 - 0.405$

$= 0.855$

5) (a)

$$P(X \le 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$$
$$= 0.05 + 0.10 + 0.15 + 0.20$$
$$= 0.50$$

(b) $$P(X \le 3 \mid X \ge 2) = \frac{P(\{X \ge 2\} \cap \{X \le 3\})}{P(\{X \ge 2\})}$$
$$= \frac{P(2 \le X \le 3)}{P(X \ge 2)}$$
$$= \frac{P(X=2) + P(X=3)}{P(X=2) + P(X=3) + P(X=4) + P(X=5)}$$
$$= \frac{0.35}{0.85} = 7/17$$

(c)
$$P(\text{bad chip} \le 3) = P(\text{bad chips} \le 3 \mid X)P(X) + P(\text{bad chips} \le 3 \mid Y)P(Y)$$
$$= 0.50 \cdot 0.25 + 0.85 \cdot 0.75$$
$$= 1/8 + 17/20 \cdot 3/4$$
$$= 61/80$$

(d) $$P(\text{from } X \mid 3 \text{ bad}) = \frac{P(3 \text{ bad} \mid \text{from } X)P(\text{from } X)}{P(3 \text{ bad})} = \frac{0.25 \cdot 0.25}{0.25 \cdot 0.25 + 0.15 \cdot 0.75}$$
$$= 0.36$$

(e) Y. Probabilities of small number of failures are large compared to X.

6) This is a quite famous problem (and a hard one) on conditional probability, Bayes Rule, Law of Total Probability. If you got this, feel good!

Key is to first define events so we can be formal:

A: event dog is in A $\quad P[A] = 0.4$

B: event dog is in B $\quad P[B] = 0.6$

L_{A_i}: event Oscar looks in A on day i. $\quad P[F_i^A \mid A] = 0.25$

D_i: event dog dies on i^{th} night $\quad P[F_i^B \mid B] = 0.15$

F_i^A: event dog is found on day i in A

F_i^B: event dog is found on day i in B

a)

$$P[F_1^A] = P[F_1^A \mid A]\,P[A] + P[F_1^A \mid B]\,P[B]$$
$$= 0.25(0.4) + 0.0$$
$$= 0.1$$

$$P[F_1^B] = P[F_1^B \mid B]\,P[B] + P[F_1^B \mid A]\,P[A]$$
$$= 0.15(0.6) = 0.09$$

Look in A!

(b) $$P[A \mid F_1^{A^c}] = \frac{P[\overline{F_1^A} \mid A]\,P[A]}{P[\overline{F_1^A}]} = \frac{0.75(0.4)}{0.9} = 1/3$$

(c) $P\left[L_A \mid \underbrace{(F_{A_1} \cap A \cap L_A) \cup (F_{B_1} \cap B \cap \bar{L}_A)}_{\text{Found on day 1}}\right]$

$$= \frac{P[F_{A_1} \cap A \cap L_A \mid L_A] \overset{0.5}{P[L_A]}}{\cancel{0.5}(0.1) + \cancel{0.5}(0.09)} = \frac{0.1}{0.19} = 0.526$$

(d) $P[A \cap \bar{F}_1^A \cap F_2^A \cap \bar{D}_1]$

$$= (1 - P[D_1])\, P[\bar{F}_1^A \cap F_2^A \mid A]\, P[A]$$

$$= \tfrac{2}{3}\ \tfrac{3}{4}\ \tfrac{1}{4}\ \tfrac{2}{5} = \tfrac{1}{20}$$

(e) $P[A \cap \bar{F}_1^A \cap F_2^A \cap D_1 \mid \bar{F}_1^A]$

$$= \frac{P[D_1 \cap A \cap F_2^A \cap \bar{F}_1^A]}{P[F_1^{A^c}]} = \frac{P[D_1]\, P[F_2^A \cap \bar{F}_1^A \mid A]\, P[A]}{P[F_1^{A^c}]}$$

$$= \frac{\tfrac{1}{3}\ \tfrac{3}{4}\ \tfrac{1}{4}\ \tfrac{2}{5}}{0.9} = \tfrac{1}{36}$$

$$\Rightarrow\ 1 - \tfrac{1}{36} = \tfrac{35}{36}$$

(f) $P[\bar{D}_1 \cap \bar{D}_2 \cap \bar{D}_3] = \tfrac{2}{3}\ \tfrac{1}{2}\ \tfrac{2}{5} = \tfrac{4}{30}$