1) \textbf{Uncountable}

We know \([0,1]\) is uncountable.

Consider any \(x \in [0,1]\) and write its binary expansion:

\[
0.a_1a_2a_3a_4\ldots
\]

and construct the set:

\[
A_x = \left\{ k \mid a_k = 1 \right\}
\]

Now \(A_x \not\subseteq \emptyset\)

and is not equal to \(A_y\) if \(x \not= y\).

Thus, for every \(x \in [0,1]\), \(\exists\) a distinct \(s_t + A_x \in Y\)

\[
= Y \text{ uncountable}
\]
2) (a) Think of a table

\[
\begin{array}{c|cc}
  & f_1 & f_2 \\
 0 & f_0(0) & f_0(1) \\
 1 & f_1(0) & f_1(1) \\
\end{array}
\]

Thus, each function can be identified by the ordered pair \((f(0), f(1)), f(i) \in \mathbb{Z}_+\)

Thus, this is \(\mathbb{Z}_+ \times \mathbb{Z}_+ \Rightarrow \text{countable}\)

(b) \((f(1), f(2), \ldots, f(n)) f(i) \in \mathbb{Z}_+\)

This is \(\mathbb{Z}_+^n\) which I claim is countable.

We know \(\mathbb{Z}_+^2\) is countable. Assume \(\mathbb{Z}_+^n\) is countable. Consider \(\mathbb{Z}_+^{n+1} = \mathbb{Z}_+^n \times \mathbb{Z}_+\). Since \(\mathbb{Z}_+^n\) is countable, it can be put 1-to-1 with \(\mathbb{Z}_+\); hence, \(\mathbb{Z}_+^n \times \mathbb{Z}_+\) can be put 1-to-1 with \(\mathbb{Z}_+^n \times \mathbb{Z}_+\), which is countable. Thus, by induction \(\mathbb{Z}_+^n\) is countable for any \(n\).

c) Let \(B_{n,i}\) ith set in \(\mathbb{B}_n\)

\[
B_{1,1} B_{1,2} B_{1,3} B_{1,4} \ldots \quad B_{1,1} B_{2,1} B_{2,2} B_{2,3} B_{2,4} \ldots \quad B_{1,1} B_{3,1} B_{3,2} B_{3,3} B_{3,4} \ldots \quad \text{Countable}
\]

Use diagonal method to list.
So a function is defined by:

\[(f(1), f(2), f(3), \ldots), f(i) \in \mathbb{N}_+\]

Consider \(E_{0,1}\). Take any \(x \in [0,1]\). We know we can write:

\[x = 0, b_1 b_2 b_3 \ldots\]

Now, \(D\) contains a function:

\[(b_1, b_2, b_3, \ldots)\]

Thus, for all \(x \in [0,1]\), there is a distinct function in \(D\).

\[\Rightarrow D\ \text{uncountable}\]

(c) \(D \subseteq E \Rightarrow E\ \text{uncountable}\)

(f)

\[(f(1), f(2), \ldots, f(N-1), 0, 0, 0, \ldots, 0)\]

\[|\{0, 1\}^{N-1}| = 2^{N-1} \Rightarrow \text{finite}\]

(g) \((f(1), f(2), \ldots, f(N-1), 1, 1, \ldots, 1)\)

\[\mathbb{Z}^{N-1} \Rightarrow \text{countable}\]
(b) \[ \mathbb{R}^{n-1} \Rightarrow \text{uncountable} \]
(recall that \( [0,1] \) uncountable, \([0,1] \subset \mathbb{R} \Rightarrow \mathbb{R} \text{ uncountable} \))

(i) \[ \forall i \neq j \]

For any \( \forall i \neq j \), there exists \( (i,j) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \),
thus, \( I \) is no larger than \( \mathbb{Z}_+ \times \mathbb{Z}_+ \Rightarrow \text{countable} \)

(iv) Let \( J_0 : \) set of all \( n \)-element subsets of \( \mathbb{Z}_+ \)
\( J_0 \) is no larger than \( \mathbb{Z}_+^n \Rightarrow \text{countable} \)

Let \( J = \bigcup_{n=1}^{\infty} J_n \) Using the same argument as (c).
\( J \) countable
3) (a) 

If $A_1, A_2, A_3, \ldots$ are in $B$, so must be 

$\overline{A_1}, \overline{A_2}, \overline{A_3}, \ldots$ (closed under complement). Thus, so is 

$\overline{A_1} \cup \overline{A_2} \cup \overline{A_3} \cup \ldots = \overline{A_1 \cap A_2 \cap A_3 \cap \ldots}$ (closed 

under countable union) and thus $\bigcap_{i=1}^{\infty} A_i$ (closed 

under complement). Thus, $B$ is closed under 

countable intersections.

For any $x \in [0,1]$

$\bigcap_{i=1}^{\infty} (x-\frac{1}{n}, x+\frac{1}{n}) \in B$

set of

just $x$-singleton

Thus, for any countable $A$, $A = \bigcup_{x \in A} x \notin B$

Thus, if $A$ is not in $B$, it must be uncountable.

(b) No. $D = [0, \frac{1}{2}]$, $\overline{D} = (\frac{1}{2}, 1]$

both uncountable

(c) 

$P(c) = 1 - P(\bar{c}) = 1 - P(D)$

$\bigcup_{x \in D} = 0$

countable 

$= 1 - \sum_{x \in D} P(\{x\})$

$= 1$
4) (a)\n
Uncountable

\[ \left| \mathcal{S} = \{ \text{Tails} \} \right|, \quad \text{Head}^{\infty} = [0, 1] \]

and \([0, 1]^{\infty}\) is uncountable.

(b) Need to use Borel-type sets. Thus, map every \(w\) to a number \(x \in [0, 1]\)

\[ x = 0.\omega, \omega_2 \omega_3 \ldots \]

where \(K\):

\[ w_k = \begin{cases} 1 & \text{flip} \quad k = \text{"heads"} \\ 0 & \text{flip} \quad k = \text{"tails"} \end{cases} \]

Now, define \(A = \) in the Borel-field where no interval \((a, b) \in A\)

corresponds to those sequences \(x\) with \(x \in (a, b)\).

Finally

\[ P((a, b)) = b - a \]

\(=\) single number \(c \in \mathbb{R}\)

(c) Equivalent to \(P(0.010101 \ldots) = 0\)

\[ x = 0.000000 \ldots \frac{1}{2} \]

\[ \{ x | 0.\omega \omega_2 \omega_3 \ldots \frac{1}{2} \} = \left( 0, \frac{1}{8} \right) \in B \]

and \(P(\left( 0, \frac{1}{8} \right)) = \frac{1}{8}\)

\[ \{ x | 0.\omega \omega_2 \omega_3 \ldots \frac{1}{2} \} \]

\[ \bigcup \left( (w_1 \frac{1}{4} + w_2 \frac{1}{4} + w_3 \frac{1}{4} + w_4 \frac{1}{4} + w_5 \frac{1}{4} + \frac{1}{8} ) \in B \right) \]

\[ w_0, w_2, w_3 \in \{ 0, 1 \} \quad \text{and} \quad P(U(\ldots)) = 2P(C, \ldots) = \frac{1}{8} \]
5) Yes. Key checks:
(a) $\mathcal{A}$ is a $\sigma$-algebra
(b) $(\mathcal{A})$ satisfies 3 axioms

(b) $\Omega = \{1, 2, 3\}$

$\mathcal{A} = \mathcal{P}(\Omega) = \{\emptyset, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

$P:\begin{align*}
p(\{1, 2\}) &= 0.4 \cdot 0.2 = 0.08 \\
p(\{1, 3\}) &= 0.4 \cdot 0.8 = 0.32 \\
p(\{2, 3\}) &= 0.6 \cdot 0.2 = 0.12 \\
p(\{1, 2, 3\}) &= 0.6 \cdot 0.8 = 0.48
\end{align*}$

For $A \in \mathcal{A}$, $P(A) = \sum_{X \in A} p(x, y)$

(c) $\Omega = \mathbb{R}$

$A = B$

Note there are three possible outcomes

$P(x = 2) = 0.4 \cdot 0.2 = 0.08$
$P(x = 4) = 0.4 \cdot 0.8 + 0.6 \cdot 0.2 = 0.44$
$P(x = 8) = 0.6 \cdot 0.8 = 0.48$

$P((a, b)) = \int_a^b (0.08 \delta(x-2) + 0.44 \delta(x-4) + 0.48 \delta(x-8)) \, dx$

(d) Yes. See (a).

(e) not in $\mathcal{A}$ = undefined