Homework #1 Solutions

ECE 603 Fall, 2016

1)

uncountable

We know $[0,1]$ is uncountable.

Consider any $x \in [0,1]$ and write its binary expansion:

$$0.a_1a_2a_3a_4\ldots$$

and construct the set:

$$A_x = \{k \mid a_k = 1\}$$

Now $A_x \in \mathcal{Y}$

and is not equal to A_y if $x \neq y$

Thus, for every $x \in [0,1]$, $\exists$ a distinct set $A_x \in \mathcal{Y}$

$\Rightarrow \mathcal{Y}$ uncountable

2) (a) Think of a table

	f_1	f_2	
0	$f_1(0)$	$f_2(0)$	...
1	$f_1(1)$	$f_2(1)$	

Thus, each function can be identified by the ordered pair

$$(f(0), f(1)), \quad f(i) \in \mathbb{Z}_+$$

Thus, this is $\mathbb{Z}_+ \times \mathbb{Z}_+ \Rightarrow$ countable

(b)

$$(f(1), f(2), \ldots, f(n)) \qquad f(i) \in \mathbb{Z}_+$$

This is $\mathbb{Z}_+^n$ which I claim is countable.

We know $\mathbb{Z}_+^2$ is countable. Assume $\mathbb{Z}_+^n$ is countable. Consider $\mathbb{Z}_+^{n+1} = \mathbb{Z}_+^n \times \mathbb{Z}_+$. Since $\mathbb{Z}_+^n$ is countable, it can be put 1-to-1 with $\mathbb{Z}_+$; hence, $\mathbb{Z}_+^n \times \mathbb{Z}_+$ can be put 1-to-1 with $\mathbb{Z}_+ \times \mathbb{Z}_+$, which is countable. Thus, by induction $\mathbb{Z}_+^n$ is countable for any n.

(c) Let $B_{n,i}$: ith set in $\mathcal{B}_n$

$B_{1,1}$	$B_{1,2}$	$B_{1,3}$	$B_{1,4}$	...
$B_{2,1}$	$B_{2,2}$	$B_{2,3}$	$B_{2,4}$	...
$B_{3,1}$	$B_{3,2}$	$B_{3,3}$	$B_{3,4}$	...
⋮				

Use diagonal method to list:

$B_{1,1}, B_{2,1}, B_{1,2}, B_{1,3}, B_{2,2}, B_{3,1}, \ldots$

Countable

(d) <u>Table</u>

	f_1	f_2
1	$f_1(1)$	$f_2(1)$
2	$f_1(2)$	$f_2(2)$
3	$f_1(3)$	$f_2(3)$
4	$\vdots$	$\vdots$
5		
6		
7		

So a function is defined by:

$$(f(1), f(2), f(3), \ldots) \qquad f(i) \in \mathbb{Z}_+$$

Consider $[0,1]$. Take any $x \in [0,1]$. We know we can write

$$x = 0.b_1 b_2 b_3 \ldots$$

Now, D contains a function

$$(b_1, b_2, b_3, \ldots)$$

Thus, for all $x \in [0,1]$, there is a distinct function in D.

$\Rightarrow$ <u>D uncountable</u>

(e) $D \subset E \Rightarrow E$ <u>uncountable</u>

(f)

$$(f(1), f(2), \ldots, f(N-1), 0, 0, \ldots, 0)$$

$$|\{0,1\}^{N-1}| = 2^{N-1} \Rightarrow \underline{\text{finite}}$$

(g) $(f(1), f(2), \ldots, f(N-1), 1, 1, \ldots, 1)$

$$\mathbb{Z}_+^{N-1} \Rightarrow \underline{\text{countable}}$$

(h)

$\mathbb{R}^{N-1} \Rightarrow$ uncountable

~~(recall that [0,1] uncountable,~~

~~$[0,1] \subset \mathbb{R}, \Rightarrow \mathbb{R}$ uncountable)~~

(i)

$\{i, j\}$ $i \neq j$

For any $\{i, j\}$, there exists $(i, j) \in \mathbb{Z}_+ \times \mathbb{Z}_+$; thus, I is no larger than $\mathbb{Z}_+ \times \mathbb{Z}_+ \Rightarrow$ countable

(j)

Let J_n: set of all n-element subsets of $\mathbb{Z}_+$

J_n is no larger than $\mathbb{Z}_+^n \Rightarrow$ countable

Let $J = \bigcup_{n=1}^{\infty} J_n$. Using the same argument as (c),

J countable

3) (a)

If $A_1, A_2, A_3, \ldots$ are in $\mathbb{B}$, so must be $\bar{A}_1, \bar{A}_2, \bar{A}_3, \ldots$ (closed under complement). Thus, so is $\bar{A}_1 \cup \bar{A}_2 \cup \bar{A}_3 \cup \ldots = \overline{A_1 \cap A_2 \cap A_3 \cap \ldots}$ (closed under countable union) and thus $\bigcap_{i=1}^{\infty} A_i$ (closed under complement). Thus, $\mathbb{B}$ is closed under countable intersections.

For any $x \in [0,1]$

$$\underbrace{\{x\}}_{\substack{\text{set of} \\ \text{just } x\text{: singleton}}} = \bigcap_{i=1}^{\infty} (x - 1/n, x + 1/n) \in \mathbb{B}$$

Thus, for any countable A, $A = \bigcup_{x \in A} \{x\} \in \mathbb{B}$

Thus, if A is not in $\mathbb{B}$, it must be uncountable

(b) No. $D = [0, 1/2]$, $\bar{D} = (1/2, 1]$

both uncountable

(c)

$$P(C) = 1 - P(\bar{C}) = 1 - P(\mathbb{Q})$$

$$= 1 - P\Big(\bigcup_{x \in \mathbb{Q}} \{x\}\Big)$$

$\mathbb{Q}$ countable

$$= 1 - \sum_{x \in \mathbb{Q}} \cancel{P(\{x\})}^{\,0}$$

$$= 1$$

4) (a)

<u>Uncountable</u>

$|S| = |\{Tail, Head\}^\infty| = |\{0,1\}^\infty|$

and $\{0,1\}^\infty$ is uncountable

(b) Need to use Borel-type sets. Thus, map every ω to a number on $[0,1]$

$$x = 0.\omega_1 \omega_2 \omega_3 \ldots$$

where $\omega_k = \begin{cases} 1, & \text{flip } k = \text{"heads"} \\ 0, & \text{flip } k = \text{"tails"} \end{cases}$

Now, define A as in the Borel field, where an interval

$$(a,b) \in A$$

corresponds to those sequences s.th. $x \in (a,b)$.

Finally

$$P((a,b)) = b-a$$

a single number $\in \mathbb{B}$

(c) • Equivalent to $P(0.010101\ldots) = 0$

• $\{x \mid 0.000\omega_4\omega_5\omega_6\ldots\} = [0, 1/8) \in \mathbb{B}$

and $P([0,1/8)) = 1/8$

• $\{x \mid 0.\omega_1\omega_2\omega_3 000\omega_7\omega_8\omega_9\ldots\}$

$= \bigcup_{\omega_1,\omega_2,\omega_3 \in \{0,1\}} (\omega_1 \cdot 1/2 + \omega_2 \cdot 1/4 + \omega_3 \cdot 1/8, \omega_1 \cdot 1/2 + \omega_2 \cdot 1/4 + \omega_3 \cdot 1/8 + 1/8) \in \mathbb{B}$

and $P(\cup(\ ,\ )) = \Sigma P((\ ,\ )) = 1/8$

5) (a) Yes. key checks: (i) $\mathcal{A}$ is a σ-algebra
(ii) $P(\cdot)$ satisfies 3 axioms

(b) $\Omega = \{(1,2), (1,4), (2,2), (2,4)\}$

$$\mathcal{A} = \mathcal{P}(\Omega) = \{\phi, \{(1,2)\}, \{(1,4)\}, \{(2,2)\}, \{(2,4)\}, \{(1,2),(1,4)\}, \{(1,2),(1,4)\}, \ldots, \Omega\}$$

P:
$$\left.\begin{aligned} P((1,2)) &= 0.4 \cdot 0.2 = 0.08 \\ P((1,4)) &= 0.4 \cdot 0.8 = 0.32 \\ P((2,2)) &= 0.6 \cdot 0.2 = 0.12 \\ P((2,4)) &= 0.6 \cdot 0.8 = 0.48 \end{aligned}\right\} \text{ all singletons}$$

For $A \in \mathcal{A}$, $P(A) = \sum_{x_i \in A} \underbrace{P(\{x_i\})}_{\text{sum probs of singletons in } A}$

(c)

$\Omega = \mathbb{R}$ $\mathcal{A} = \mathbb{B}$

Note there are three possible outcomes

$$\begin{aligned} P(X=2) &= 0.4 \cdot 0.2 = 0.08 \\ P(X=4) &= 0.4 \cdot 0.8 + 0.6 \cdot 0.2 = 0.44 \\ P(X=8) &= 0.6 \cdot 0.8 = 0.48 \end{aligned}$$

$$P((a,b)) = \int_a^b \big(0.08\,\delta(x-2) + 0.44\,\delta(x-4) + 0.48\,\delta(x-8)\big)\,dx$$

(d) Yes. See (a).

(e) not in $\mathcal{A}$ $\Rightarrow$ <u>undefined</u>